Polynomial decay of correlations for pseudo-Anosov diffeomorphisms

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Abstract

We give a construction of a smooth realization of a pseudo-Anosov diffeomorphism of a Riemannian surface, and show that it admits a unique SRB measure with polynomial decay of correlations, large deviations, and the central limit theorem. The construction begins with a linear pseudo-Anosov diffeomorphism whose singularities are fixed points. Near the singularities, the trajectories are slowed down, and then the map is conjugated with a homeomorphism that pushes mass away from the origin. The resulting map is a $C^{2+\beta}$ diffeomorphism with nonzero Lyapnuov exponents, which is topologically conjugate to the original pseudo-Anosov map. We also show that with respect to its SRB measure, the map has polynomial upper and lower large deviation bounds and the central limit theorem; and has a unique measure of maximal entropy with respect to which the map is Bernoulli, has nonzero Lyapunov exponents, and has both exponential decay of correlations and the central limit theorem with respect to Hölder observables.

Keywords: Smooth dynamics, low-dimensional dynamics, decay of correlations, thermodynamic formalism

MSC Classification: 37D35 , 37D25 , 37D05 , 37E30 , 37A25 , 37A05

1 Introduction

In [1], A. Katok introduced a C^∞ area-preserving diffeomorphism of $\mathbb{T}^2$ with *strict nonuniform hyperbolicity:* the map $G_{\mathbb{T}^2} : \mathbb{T}^2 \to \mathbb{T}^2$ (known as the *Katok map of the torus*) has nonzero Lyapunov exponents Lebesgue-almost everywhere, but admits a singularity at the origin at which the differential $dG_{\mathbb{T}^2}$ is the identity. As a consequence, there are trajectories admitting zero Lyapunov exponents, and so the Lyapunov exponents of $G_{\mathbb{T}^2}$ come arbitrarily close to 0 over $\mathbb{T}^2$ (this is the sense in which the nonuniform hyperbolicity is strict). The map $G_{\mathbb{T}^2}$ is a Bernoulli automorphism (meaning $G_{\mathbb{T}^2}$ is measurably isomorphic to a Bernoulli shift), and is topologically conjugate to a linear Anosov diffeomorphism of $\mathbb{T}^2$. Then in [2], A. Katok and M. Gerber extended the construction of the Katok map to any compact Riemannian surface, presenting a wide family of area-preserving C^∞ diffeomorphisms that are both strictly nonuniformly hyperbolic and Bernoulli. However, unlike the Katok map $G_{\mathbb{T}^2}$, the Bernoulli diffeomorphisms in [2] are not topologically conjugate to an Anosov map. Indeed, Anosov diffeomorphisms on surfaces admit global stable and unstable 1-dimensional foliations, and no surface with genus $\neq 1$ admits 1-dimensional foliations, so the only surface that admits Anosov diffeomorphisms is the torus $\mathbb{T}^2$. So instead of being conjugate to an Anosov diffeomorphism, the nonuniformly hyperbolic diffeomorphisms in [2] are topologically conjugate to the broader class of *pseudo-Anosov diffeomorphisms.*

Pseudo-Anosov diffeomorphisms were introduced by W. Thurston in [3] as a generalization of Anosov diffeomorphisms: rather than admitting global stable and unstable submanifolds, pseudo-Anosov maps admit stable and unstable *foliations with singularities*, for which there are a finite number of singularities where multiple leaves of the foliations meet (see Section 2.1). In the theory of mapping class groups, pseudo-Anosov diffeomorphisms play a role in the *Nielsen-Thurston classification* of surface homeomorphisms:

Theorem *Let M be a compact orientable surface, and let $f : M \to M$ be a homeomorphism. Then f is isotopic to a homeomorphism $F : M \to M$ satisfying exactly one of the following three conditions:*

- *F is a rotation: there is an integer $n \geq 1$ for which $F^n = \mathrm{id}$.*
- *F is a Dehn twist: there is a closed curve that F leaves invariant.*
- *F is pseudo-Anosov.*

The construction of the Katok map $G_{\mathbb{T}^2}$ and the smooth realizations of the pseudo-Anosov maps in [2] both use a similar *slow-down procedure.* In the case of the Katok map $G_{\mathbb{T}^2}$, the construction starts with a linear hyperbolic toral automorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ induced by a matrix $A \in \mathrm{SL}(2, \mathbb{Z})$. The map f is then written in coordinates near the origin as the time-1 map of the flow induced by the system of ODEs

$$\dot{s}_1 = s_1 \log \lambda, \quad \dot{s}_2 = -s_2 \log \lambda, \tag{1}$$

where (s_1, s_2) are the coordinates induced by the eigenvectors of A with eigenvalues $\lambda > 1$ and $\lambda^{-1} < 1$, respectively. The trajectories of the flow near the origin are then "slowed down" by multiplying (1) by a function $\psi(u) = Cu^\alpha$, for a fixed $\alpha > 0$, producing the system of ODEs:

$$\dot{s}_1 = \psi(s_1^2 + s_2^2) s_1 \log \lambda, \quad \dot{s}_2 = -\psi(s_1^2 + s_2^2) s_2 \log \lambda. \tag{2}$$

The time-1 map of the resulting flow has a differential at the origin equal to the identity. Finally, the new time-1 map is conjugated with a homeomorphism that makes the resulting map area-preserving; the final map is $G_{\mathbb{T}^2}$.

In constructing the nonuniformly hyperbolic surface diffeomorphisms in [2], Gerber and Katok begin with a pseudo-Anosov homeomorphism $f : M \to M$ whose singularities are fixed points, construct a continuous vector field in coordinates around each singularity whose time-1 map is f, and similarly slow down the vector field trajectories to produce a new time-1 diffeomorphism $g : M \to M$ whose differential at the singularities is the identity. The slow-down procedures used to construct the pseudo-Anosov maps in [2] and the Katok map $G_{\mathbb{T}^2}$ in [1] are similar, but the slow-down procedure used to construct $G_{\mathbb{T}^2}$ in [1] is presented very generally, giving great flexibility with the rate at which the flow trajectories slow down. In contrast, for the pseudo-Anosov maps in [2], a specific slow-down rate α is given. The resulting slowed-down diffeomorphism preserves the area given by the coordinates around each singularity, so no conjugating map is required to further make the diffeomorphism area-preserving.

In [4], Y. Pesin, S. Senti, and F. Shahidi showed that the Katok map $G_{\mathbb{T}^2}$ has a range of thermodynamic properties. They demonstrate that the unique SRB measure of $G_{\mathbb{T}^2}$ has polynomial decay of correlations (rate of mixing), the central limit theorem, and polynomial large deviations with respect to Hölder observables. They also show (also in [5] with K. Zhang) that $G_{\mathbb{T}^2}$ has a unique measure of maximal entropy, with respect to which $G_{\mathbb{T}^2}$ has exponential decay of correlations and the central limit theorem. Furthermore, as their main result, they give a construction of a nonuniformly hyperbolic diffeomorphism of any compact surface that comes from gluing the singularity of $G_{\mathbb{T}^2}$ to the surface, and the resulting diffeomorphism has the same thermodynamic properties as the Katok map.

The goal of this paper is to generalize the smooth pseudo-Anosov realization of [2], with the purpose of producing a strictly nonuniformly hyperbolic diffeomorhism that enjoys the same properties as the Katok map described in [4]. The main result of [4] states that any compact surface admits a nonuniformly hyperbolic diffeomorphism with polynomial decay of correlations, the central limit theorem, and large deviations; we provide an alternative construction to the one in [4]. Unlike the construction in [4], which begins with the Katok map and uses a sequence of maps $\mathbb{T}^2 \to \mathbb{S}^2 \to \mathbb{D}^2 \to M$ to produce a semi-conjugacy between $G_{\mathbb{T}^2}$ and a map of M (where $\mathbb{S}^2$ and $\mathbb{D}^2$ are the 2-sphere and the 2-disk, respectively), our construction directly produces a diffeomorphism that is topologically conjugate to a given pseudo-Anosov homeomorphism $f : M \to M$.

The techniques in [4] are based on modeling the Katok map with a *Young tower*, a symbolic representation of hyperbolic maps by a tower whose base is conjugate to a two-sided countable-state Bernoulli shift, originally introduced in [6]. A Young tower for a map $f : M \to M$ is constructed by considering a suitable subset $\Lambda \subset M$, a countable collection $\{\Lambda_i\}_{i\geq 1}$ of subsets of Λ, and a function $\tau : \Lambda \to \mathbb{N}$ constant on each Λ_i for which $f^{\tau(x)}(x) \in \Lambda$ for each $x \in \Lambda$. The function τ is called the *return time* or *inducing time* of the tower, and Λ is the *base* of the tower (see Section 5 for details). In [7], we used Young towers to show that the smooth realizations of pseudo-Anosov maps in [2] have a unique measure of maximal entropy with exponential decay of correlations and the central limit theorem with respect to Hölder potentials. Furthermore, there is a $t_0 < 0$ for which for $t \in (t_0, 1)$, the geometric t-potentials $\varphi_t(x) = -t \log \left| dg|_{E^u(x)} \right|$ admit unique equilibrium states with exponential decay of correlations and the central limit theorem; meanwhile, the geometric potential $\varphi_1(x) = -\log \left| dg|_{E^u(x)} \right|$ has two classes of equilibrium states: a unique SRB measure, and convex combinations of Dirac masses at the singularities. These results mirror the results on the Katok map presented in [5].

The main technical challenge of this work is constructing a pseudo-Anosov smooth model for which the strategies in [4] can be applied. To prove polynomial decay of correlations for $G_{\mathbb{T}^2}$ in [4], the authors require that the slow-down exponent $\alpha > 0$ defining $\psi(u)$ in (2) satisfies $\alpha < 1/3$. By contrast, the slow-down rate of the pseudo-Anosov smooth realizations in [2] is specifically chosen to be $\alpha = \frac{p-2}{p}$, where $p \geq 3$ is the number of *prongs of the singularity* (see Section 2.1). So the slow-down exponent $\frac{p-2}{p}$ falls outside of the range for which the arguments in [4] can be directly applied. One of the goals of this paper is to adapt the construction in [2] in a way that produces a nonuniformly hyperbolic surface diffeomorphism with a variety of topological and ergodic properties. In particular, we produce a nonuniformly hyperbolic $C^{2+\beta}$ Bernoulli diffeomorphism that is topologically conjugate to a pseudo-Anosov homeomorphism, and whose unique SRB measure has polynomial decay of correlations, the central limit theorem, and polynomial large deviations.

Young towers have been used frequently to show that nonuniformly hyperbolic and nonuniformly expanding dynamical systems have different statistical properties. In [6, 8], L.-S. Young showed that for a large class of nonuniformly expanding and nonuniformly hyperbolic dynamical systems $f : M \to M$, the *tail of the return time*

$$\mu\{x \in \Lambda : \tau(x) > n\},$$

where μ is the SRB measure of f, decays with n at the same or similar asymptotic rate as the correlations for a family of observables. Since then, there have been several examples of nonuniformly expanding and nonuniformly hyperbolic systems with a symbolic representation (not necessarily a Young tower) for which the decay of the tail of the return time is used to prove subexponential decay of correlations. For example, O. Sarig and S. Gouëzel [9, 10] show that for Markov maps whose tails decay polynomially, if the observables are supported on the inducing domain, then the correlations for those observables admit both an *upper* and *lower* polynomial bound on the decay rate, and these bounds are the same up to a constant. These results show that *Manneville-Pomeau* maps of the interval [11, 12] have polynomial decay of correlations for observables supported on the entire interval. Furthermore, in [9], Gouëzel also gives an *upper* polynomial bound on decay of correlations for dynamical systems admitting Young towers, extending a version of the result on Markov maps to the two-dimensional setting. More recently, H. Bruin and D. Terhesiu showed in [13] that nonuniformly expanding Gibbs-Markov maps with a suitable "reinduce time" have upper and lower bounds on correlations with the same decay function; this decay may be polynomial or (sub- or super-)exponential depending on the tails of the map. Furthermore, H. Bruin, I. Melbourne, and D. Terhesiu showed in [14] that maps admitting a *Chernov-Markarian-Zhang* structure (roughly speaking, a strengthened Young structure in which the base of the tower admits a further Young structure) have polynomial upper and lower bounds on decay of correlations with the same asymptotic rate up to a constant. Maps satisfying these conditions include Hu-Vaienti maps and several billiards systems. For additional examples of how Young structure is used to study rates of mixing in nonuniformly hyperbolic and nonuniformly expanding systems, see the examples in [15].

We remark that the results in [9, 13, 14] require sharp upper and lower bounds on the decay of the tails, i.e., an upper and lower bound of the same order. In our setting, however, we produce upper and lower bounds on the decay of the tails, but they are of different orders (Lemmas 7.4 and 8.3). This comes from the difficulty in estimating the time a typical orbit spends near singularities. It may be possible to improve our estimate on the decay of the tail by improving our analysis of the behavior near singularities; at this time, we do not know how such improvements would be carried out. Furthermore, [9] produces sharp estimates on the decay of correlations for observables supported on the base of the Young tower; our results extend these conditions by considering potentials supported on an arbitrarily large subset of the manifold.

Additionally, the decay of the tail of the return time has been used to study the decay of large deviations. I. Melbourne and M. Nicol showed in [16] that if the tail of the return time for a nonuniformly expanding map decays at a polynomial rate, then the map has polynomial large deviations, which decays at a similar (but not equal) asymptotic rate as the tail. Similar results have been obtained for nonuniformly hyperbolic maps, with the additional assumption that the induced map $f^\tau : \Lambda \to \Lambda$ has exponential contraction in stable manifolds [16, 17]. Lastly, in [18], I. Melbourne improved the result in [16] for nonuniformly expanding maps, showing that nonuniformly expanding maps with polynomial decay of correlations for L^∞ observables have polynomial large deviations with the *same* asymptotic rate of decay as the correlation functions. This does not extend to the setting of nonuniformly hyperbolic (invertible) maps with polynomial decay of correlations only for Hölder functions, as opposed to all L^∞ functions, and so we cannot show in our result that the large deviation asymptotic behavior is equal to the asymptotic behavior of the decay of correlations.

We end by noting that many examples of strictly nonuniformly hyperbolic dynamical systems are constructed by introducing a *neutral fixed point*, which is a fixed point whose differential is the identity (as is done for the Katok map and for the pseudo-Anosov smooth realizations). In many of these cases, the resulting map also has a unique SRB measure with polynomial decay of correlations. In the one-dimensional category, the Manneville-Pomeau map is a nonuniformly expanding map with a fixed point at which the derivative is 1 [12]; as mentioned previously, it has been shown that the Manneville-Pomeau map and other related one-dimensional transformations have a unique invariant measure absolutely continuous to Lebesgue, with respect to which the map admits polynomial decay of correlations [9, 11, 19]. Additionally, L.S.-Young introduced in [8] an expanding homeomorphism of the circle with an indifferent fixed point at the origin that has polynomial decay of correlations for its unique SRB measure. In the two-dimensional category, in addition to the examples of diffeomorphisms with indifferent fixed points considered in [4], H. Hu constructed a different class of diffeomorphisms with indifferent fixed points called "almost Anosov" maps [20], and demonstrated with X. Zhang in [21] that these diffeomorphisms also have polynomial decay of correlations. Almost Anosov maps are similar to the pseudo-Anosov smooth models considered in this paper, but assume a stronger condition on the cone fields than is enjoyed by pseudo-Anosov maps. In the category of dissipative diffeomorphisms with hyperbolic attractors, J. Alves and V. Pinheiro constructed in [22] a nonuniformly hyperbolic solenoid map on the solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ with an indifferent fixed point. They showed that this map admits a Young structure, and used this Young structure to show that the "solenoid with intermittency" has polynomial decay of correlations. Finally, S. Burgos in [23] considered a dissipative dynamical system with a uniformly hyperbolic attractor, which has a hyperbolic fixed point that can be slowed down to an indifferent fixed point (following the procedure in [24]). Using techniques from [4] and [25], Burgos showed that this map's unique SRB measure has polynomial decay of correlations. This dissipative map studied in [23–25] is another example of a strictly nonuniformly hyperbolic diffeomorphism whose indifferent fixed point is induced from the slow-down procedure. The pseudo-Anosov smooth realizations in [2, 7] are constructed using a similar slow-down procedure; however, rather than starting from a dissipative map, our construction begins with a homeomorphism that preserves a measure that is absolutely continuous with respect to Lebesgue, and so the final map is area-preserving with respect to a *smooth* measure. Other examples of nonuniformly hyperbolic dynamical systems constructed using the slow-down procedure are found in [26, 27].

This paper is organized as follows. In Section 2, we introduce the preliminary definitions needed for our main results; in particular, pseudo-Anosov homeomorphisms and the relevant statistical properties. In Section 3, we state our main results. The construction of the diffeomorphism g is given in Section 4, and we show in Section 5 that the resulting map has a Young tower. In Section 6, we study different technical estimates near the singularities of g. The tail of the return time is estimated in Sections 7 and 8, and in Section 9, we prove the main result.

2 Preliminaries

2.1 Pseudo-Anosov maps

Before we define pseudo-Anosov homeomorphisms and construct their smooth realizations, we briefly discuss measured foliations with singularities. Our exposition is adapted from the presentation in [28], Section 6.4. For the reader's convenience, we have restated their exposition here. Also see Section 2 of [7].

Definition 2.1 A *measured foliation with singularities* is a triple $(\mathcal{F}, S, \nu)$, where:

- $S = \{x_1, \ldots, x_m\}$ is a finite set of points in M, called *singularities*;
- $\mathcal{F} = \tilde{\mathcal{F}} \uplus \mathcal{S}$ is a partition of M, where $\mathcal{S}$ is a partition of S into points and $\tilde{\mathcal{F}}$ is a smooth foliation of $M \setminus S$;

- ν is a *transverse measure*; in other words, ν is a measure defined on each curve on M transverse to the leaves of $\tilde{\mathcal{F}}$;

and the triple satisfies the following properties:

1. There is a finite atlas of C^∞ charts $\phi_k : U_k \to \mathbb{C}$ for $k = 1, \ldots, \ell$, $\ell \geq m$.
2. For each $k = 1, \ldots, m$, there is a number $p = p(k) \geq 3$ of elements of $\tilde{\mathcal{F}}$ meeting at $x_k \in S$ (these elements are called *prongs* of x_k) such that:
   (a) $\phi_k(x_k) = 0$ and $\phi_k(U_k) = D_{a_k} := \{z \in \mathbb{C} : |z| \leq a_k\}$ for some $a_k > 0$;
   (b) if $C \in \tilde{\mathcal{F}}$, then the components of $C \cap U_k$ are mapped by ϕ_k to sets of the form
   $$\left\{z \in \mathbb{C} : \operatorname{Im}\left(z^{p/2}\right) = \text{constant}\right\} \cap \phi_k(U_k);$$
   (c) the measure $\nu|U_k$ is the pullback under ϕ_k of
   $$\left|\operatorname{Im}\left(dz^{p/2}\right)\right| = \left|\operatorname{Im}\left(z^{(p-2)/2}dz\right)\right|.$$
3. For each $k > m$, we have:
   (a) $\phi_k(U_k) = (0, b_k) \times (0, c_k) \subset \mathbb{R}^2 \approx \mathbb{C}$ for some $b_k, c_k > 0$;
   (b) If $C \in \tilde{\mathcal{F}}$, then components of $C \cap U_k$ are mapped by ϕ_k to lines of the form
   $$\{z \in \mathbb{C} : \operatorname{Im} z = \text{constant}\} \cap \phi_k(U_k).$$
   (c) The measure $\nu|U_k$ is given by the pullback of $|\operatorname{Im} dz|$ under ϕ_k.

A singularity with $p = 3$ prongs is shown in Figure 1.

Remark 2.2 Henceforth, we refer to the C^∞ curves that are elements of $\mathcal{F}$ as "leaves (of the foliation)"; in particular, despite the technical fact that the singleton sets of singularities $\{x_1\}, \ldots, \{x_k\}$ are elements of $\mathcal{F}$, we do not refer to these points when we refer to "leaves of the foliation".

Definition 2.3 A surface homeomorphism f of a manifold M is *pseudo-Anosov* if there are measured foliations with singularities $(\mathcal{F}^s, S, \nu^s)$ and $(\mathcal{F}^u, S, \nu^u)$ (with the same finite set of singularities $S = \{x_1, \ldots, x_m\}$) and an atlas of C^∞ charts $\phi_k : U_k \to \mathbb{C}$ for $k = 1, \ldots, \ell$, $\ell > m$, satisfying the following properties:

1. f is differentiable, except on S.
2. For each $x_k \in S$, $\mathcal{F}^s$ and $\mathcal{F}^u$ have the same number $p(k)$ of prongs at x_k.
3. The leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ intersect transversally at nonsingular points.
4. Both measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ are f-invariant.
5. There is a constant $\lambda > 1$ such that
   $$f(\mathcal{F}^s, \nu^s) = (\mathcal{F}^s, \nu^s/\lambda) \quad \text{and} \quad f(\mathcal{F}^u, \nu^u) = (\mathcal{F}^u, \lambda\nu^u).$$
6. For each $k = 1, \ldots, m$, we have $x_k \in U_k$, and $\phi_k : U_k \to \mathbb{C}$ satisfies:

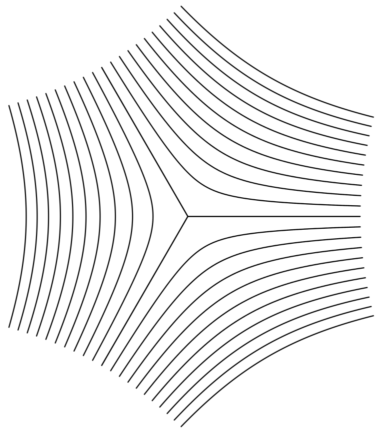

Fig. 1 A 3-pronged singularity of a measured foliation with singularities.

(a) $\phi_k(x_k) = 0$ and $\phi_k(U_k) = D_{a_k}$ for some $a_k > 0$;
(b) if C is a curve leaf in $\mathcal{F}^s$, then the components of $C \cap U_k$ are mapped by ϕ_k to sets of the form

$$\left\{z \in \mathbb{C} : \mathrm{Re}\left(z^{p/2}\right) = \text{constant}\right\} \cap D_{a_k};$$

(c) if C is a curve leaf in $\mathcal{F}^u$, then the components of $C \cap U_k$ are mapped by ϕ_k to sets of the form

$$\left\{z \in \mathbb{C} : \mathrm{Im}\left(z^{p/2}\right) = \text{constant}\right\} \cap D_{a_k};$$

(d) the measures $\nu^s|U_k$ and $\nu^u|U_k$ are given by the pullbacks of

$$\left|\mathrm{Re}\left(dz^{p/2}\right)\right| = \left|\mathrm{Re}\left(z^{(p-2)/2}dx\right)\right|$$

and

$$\left|\mathrm{Im}\left(dz^{p/2}\right)\right| = \left|\mathrm{Im}\left(z^{(p-2)/2}dx\right)\right|$$

under ϕ_k, respectively.

7. For each $k > m$, we have:
   (a) $\phi_k(U_k) = (0, b_k) \times (0, c_k) \subset \mathbb{R}^2 \approx \mathbb{C}$ for some $b_k, c_k > 0$;
   (b) If C is a curve leaf in $\mathcal{F}^s$, then components of $C \cap U_k$ are mapped by ϕ_k to lines of the form

$$\{z \in \mathbb{C} : \mathrm{Re}\, z = \text{constant}\} \cap \phi_k(U_k);$$

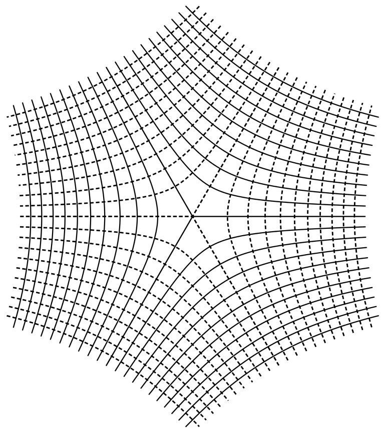

Fig. 2 A singular neighborhood with a 3-pronged singularity. The solid lines and broken lines respectively represent the stable and unstable foliations $\mathcal{F}^s$ and $\mathcal{F}^u$, for example.

(c) If C is a curve leaf in $\mathcal{F}^u$, then components of $C \cap U_k$ are mapped by ϕ_k to lines of the form

$$\{z \in \mathbb{C} : \operatorname{Im} z = \text{constant}\} \cap \phi_k(U_k);$$

(d) the measures $\nu^s|U_k$ and $\nu^u|U_k$ are given by the pullbacks of $|\operatorname{Re} dz|$ and $|\operatorname{Im} dz|$ under ϕ_k, respectively.

For $k = 1, \dots, m$, we call the neighborhood $U_k \subset M$ described in part (6) of this definition a *singular neighborhood*, and for $k > m$, we call U_k a *regular neighborhood*. (See Figure 2.)

See Remarks 2.3 and 2.5 of [7] for a discussion on some of the technical intuition behind measured foliations and pseudo-Anosov homeomorphisms.

Remark 2.4 In the literature, maps defined in this way are often referred to as "pseudo-Anosov diffeomorphisms", or "linear pseudo-Anosov diffeomorphisms", despite the technical fact that these maps are not C^1 at the singularities. Indeed, in [2], it is shown that a pseudo-Anosov homeomorphism f of the kind described here is essentially non-smooth at the singularities. To distinguish these types of systems, we reserve the word "diffeomorphisms" for maps that are smooth at every point in M, including the singularities, and refer to the maps defined in Definition 2.3 as "pseudo-Anosov homeomorphisms".

Proposition 2.5 *Let $f : M \to M$ be a pseudo-Anosov homeomorphism. For $x \in M \setminus S$, the tangent space decomposes as a direct sum $T_xM = T_x\mathcal{F}^s(x) \oplus T_x\mathcal{F}^u(x)$, where $\mathcal{F}^s(x)$ and $\mathcal{F}^u(x)$ represent the curve containing x in the respective foliation. In these coordinates, the differential of f has the diagonal form*

$$Df_x(\xi^s, \xi^u) = \left(\xi^s/\lambda, \lambda\xi^u\right),$$

where ξ^s and ξ^u are nonzero vectors in $T_x\mathcal{F}^s(x)$ and $T_x\mathcal{F}^u(x)$, and λ is the dilation factor for f.

Proof This follows immediately from the definition of pseudo-Anosov diffeomorphisms after a calculation in coordinates (see Remark 2.5 of [7]). □

Proposition 2.6 ([29]) *A pseudo-Anosov surface homeomorphism $f : M \to M$ preserves a smooth probability measure ν defined locally as the product of ν^s on $\mathcal{F}^u$-leaves with ν^u on $\mathcal{F}^s$-leaves. In any coordinate chart of M, this probability measure ν has a density with respect to the measure induced by the Lebesgue measure on $\mathbb{R}^2$, and this density vanishes at singularities.*

Proposition 2.7 ([29]) *Every pseudo-Anosov homeomorphism of a surface M admits a finite Markov partition of arbitrarily small diameter. The system (M, f, ν) has the Bernoulli property via the symbolic representation for this Markov partition (see Definition 2.8 below), where ν is the measure in the preceding proposition.*

2.2 Ergodic properties

For the reader's convenience, we describe here the thermodynamic and ergodic properties that we will refer to throughout the paper. Throughout the following, $T : X \to X$ will be a measurable and invertible transformation preserving a measure μ on X.

Definition 2.8 The transformation (T, μ) has the *Bernoulli property* if there is a Lebesgue space (Y, ν) for which (T, μ) is metrically isomorphic to the corresponding Bernoulli shift $\sigma : Y^{\mathbb{Z}} \to Y^{\mathbb{Z}}$, where $Y^{\mathbb{Z}}$ is endowed with the measure $\nu^{\otimes \mathbb{Z}}$.

Definition 2.9 Suppose $\mathcal{H}_1$ and $\mathcal{H}_2$ are two classes of real-valued functions on X (also called *observables* on (T, μ)). For $h_1 \in \mathcal{H}_1$ and $h_2 \in \mathcal{H}_2$, the n^{th} *correlation* between the two observables is

$$\mathrm{Cor}_n(h_1, h_2) := \int h_1(T^n(x))h_2(x)d\mu(x) - \int h_1\, d\mu \int h_2\, d\mu.$$

The transformation (T, μ) has *exponential decay of correlations* if there is a constant $\gamma_0 > 0$ for which for any $h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2$,

$$|\mathrm{Cor}_n(h_1, h_2)| \le C_0 e^{-\gamma_0 n},$$

where $C_0 = C_0(h_1, h_2)$ is independent of n.

The transformation T has *polynomial upper* or *lower bound on correlations* with respect to $\mathcal{H}_1$ and $\mathcal{H}_2$ if, respectively, there is a number $\gamma_1 > 0$ for which for any $h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2$,

$$|\mathrm{Cor}_n(h_1, h_2)| \le C_1 n^{-\gamma_1};$$

or, if there is a number $\gamma_2 > 0$ for which for any $h_1 \in \mathcal{H}_1, h_2 \in \mathcal{H}_2$,

$$|\mathrm{Cor}_n(h_1, h_2)| \ge C_2 n^{-\gamma_2},$$

where in each case C_1 and C_2 are constants independent of n (but may depend on h_1, h_2).

Definition 2.10 The system (T, μ) satisfies the *central limit theorem (CLT)* with respect to a class $\mathcal{H}$ of observables if there is a $\sigma > 0$ such that for any $h \in \mathcal{H}$ with $\int h\, d\mu = 0$, we have

$$\mu\left\{x \in X : \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left(h(T^i(x)) - \int h\, d\mu\right) < t\right\} \xrightarrow{n\to\infty} \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\tau^2/2\sigma^2}\, d\tau.$$

Definition 2.11 The transformation (T, μ) has *polynomial large deviation* with respect to a class $\mathcal{H}$ of observables on X if there is a $\beta > 0$ such that for any $h \in \mathcal{H}$, any $\varepsilon > 0$, and any sufficiently large $n > 0$, we have

$$\mu\left\{x \in X : \left|\frac{1}{n}\sum_{i=0}^{n-1} h\left(T^i(x)\right) - \int h\, d\mu\right| > \varepsilon\right\} < Kn^{-\beta},$$

where $K = K(h, \varepsilon) > 0$ is a constant independent of n.

3 Main results

We now state our main result. The exponents γ_1, γ_2 for the polynomial decay depend on parameters related to the slow-down procedure for the pseudo-Anosov homeomorphism f. The specific values are $\gamma_1 = \gamma' - 2$ and $\gamma_2 = \gamma - 2$, where γ and γ' are given in (23) (see also Remark 6.4).

Theorem 3.1 *Let $f : M \to M$ be a pseudo-Anosov homeomorphism of a compact orientable Riemannian surface M. There is a $\beta > 0$ and a $C^{2+\beta}$ diffeomorphism $g : M \to M$ that is topologically conjugate and C^0-close to f. Furthermore, the map g also satisfies the following properties:*

1. *g preserves a probability measure μ_1 that is equivalent to the Riemannian area of M.*
2. *g has nonzero Lyapunov exponents at μ_1-a.e. x.*
3. *g has the Bernoulli property with respect to μ_1.*
4. *g has polynomial upper and lower bounds on the correlations with respect to μ_1 and the set of η-Hölder continuous functions for any $\eta > 0$. More precisely:*
   (a) *for any $h_i \in C^\eta$, $i = 1, 2$,*

   $$\mathrm{Cor}_n(h_1, h_2)| \le C_1 n^{-\gamma_1},$$

   where $C_1 = C_1(h_1, h_2)$;
   (b) *there is a nested sequence of subsets $\{M_j\}_{j\ge1}$ that exhausts M for which if $h_1, h_2 \in C^\eta$ are such that $\int h_1\, d\mu_1 \int h_2\, d\mu_2 > 0$ and $\mathrm{supp}(h_i) \subset M_j$ for some j, for $i = 1, 2$,*

   $$|\mathrm{Cor}_n(h_1, h_2)| \ge C_2 n^{-\gamma_2},$$

   where $C_2 = C_2(h_1, h_2)$.
5. *g satisfies the CLT with respect to the class of observables $C^\eta_0 := \{h \in C^\eta : \int h\, d\mu_1 = 0\}$ for any $\eta > 0$, with $\sigma = \sigma(h)$ given by*

$$\sigma^2 = -\int h^2\, d\mu_1 + 2\sum_{n=0}^{\infty}\int h \cdot (h \circ g^n)\, d\mu_1$$

where $\sigma > 0$ iff h is not cohomologous to zero (i.e., $h \circ g \neq h' \circ g - h'$ for any $h' \in C^\eta$).

6. *g has polynomial large deviations with respect to the class of Hölder observables. Specifically, for any $a > 0$, any $\eta > 0$, any $h \in C^\eta$, and any $\varepsilon > 0$, there is a constant $C_{h,\varepsilon}$ depending continuously on ε and $\|h\|_{C^\eta}$ for which*

$$\mu_1 \left\{ \left| \frac{1}{n} \sum_{k=0}^{n-1} h(g^k(x)) - \int h \, d\mu_1 \right| > \varepsilon \right\} \leq C_{h,\varepsilon} n^{-(\gamma_1 - a)} \quad \text{for all } n \geq 1.$$

Furthermore, for any $a > 0$, there is an open and dense subset of Hölder observables h for which for all sufficiently small $\varepsilon > 0$,

$$n^{-(\gamma_1 + a)} < \mu_1 \left(\left| \frac{1}{n} \sum_{j=0}^{n-1} h(g^j(x)) - \int h \, d\mu_1 \right| > \varepsilon \right)$$

for infinitely many n;

7. *g has a unique measure of maximal entropy, with respect to which g has the Bernoulli property, nonzero Lyapunov exponents almost everywhere, exponential decay of correlations, and the Central Limit Theorem with respect to Hölder observables.*

We list how the different constants will be chosen:

- β will be given by $\frac{2}{1-\alpha} - \left\lfloor \frac{2}{1-\alpha} \right\rfloor$, where α is a slow-down exponent used in defining g.
- The exponents γ_1 and γ_2 are given by $\gamma_1 = \gamma' - 2$ and $\gamma_2 = \gamma - 2$, where γ' and γ are defined in (23) (see also Remark 6.4). Note in particular that $\gamma > \gamma' > 3$ when $\alpha < 1/6$ by (23).

4 Construction of the smooth pseudo-Anosov model

As shown in Section 2.4 of [2], pseudo-Anosov homeomorphisms as we've defined them are not smooth at the singularities. We construct a smooth realization of the pseudo-Anosov map, adapted from the procedures in both [2] and [1]. The resulting map $g : M \to M$ will be a $C^{2+\varepsilon}$ diffeomorphism whose differential at the singularities is the identity.

Before proceeding with the construction, we point out that some literature refers to the maps defined in Definition 2.3 as "pseudo-Anosov diffeomorphisms", despite the fact that these maps are not differentiable at the singularities. To avoid any confusion, we reserve the word "diffeomorphism" only for those maps that are differentiable on all of M, and use the phrase "pseudo-Anosov homeomorphism" for the maps described in Definition 2.3.

4.1 Construction of g

Let x_k be a singularity of f, let $p = p(x_k)$ be the number of prongs at this singularity, and let $\phi_k : U_k \to \mathbb{C}$ be the chart described in part (6) of Definition (2.3). The *stable* and *unstable prongs* at x_k are the leaves P^s_{kj} and P^u_{kj}, $j = 0, \ldots, p-1$ of $\mathcal{F}^s$ and $\mathcal{F}^u$, respectively, whose endpoints meet at x_k. Locally, they are given by:

$$P^s_{kj} = \phi_k^{-1}\left\{\rho e^{i\tau} : 0 \le \rho < a_k,\ \tau = \frac{2j+1}{p}\pi\right\},$$
$$\text{and}\quad P^u_{kj} = \phi_k^{-1}\left\{\rho e^{i\tau} : 0 \le \rho < a_k,\ \tau = \frac{2j}{p}\pi\right\}.$$

Since $f : M \to M$ is a homeomorphism, f permutes the singularities. Therefore, after taking a suitable iterate, assume the singularities are fixed points, and moreover, assume $f(P^s_{kj}) \subseteq P^s_{kj}$ for all $j = 0, \ldots, p-1$. Furthermore, we define the *stable* and *unstable sectors* at x_k to be the regions in U_k bounded by the stable (resp. unstable) prongs:

$$S^s_{kj} = \phi_k^{-1}\left\{\rho e^{i\tau} : 0 \le \rho < a_k,\ \frac{2j-1}{p}\pi \le \tau \le \frac{2j+1}{p}\pi\right\},$$
$$\text{and}\quad S^u_{kj} = \phi_k^{-1}\left\{\rho e^{i\tau} : 0 \le \rho < a_k,\ \frac{2j}{p}\pi \le \tau \le \frac{2j+2}{p}\pi\right\}.$$

Assume, after taking a suitable iterate, that $f(S^u_{kj}) \subset S^u_{kj}$ and $f(S^s_{kj}) \subset S^s_{kj}$.

Our strategy will be to apply a "slow-down" of the trajectories in each stable sector S^s_{kj}, followed by a change of coordinates ensuring the resulting diffeomorphism g preserves the measure induced by a convenient Riemannian metric.

Let $F : \mathbb{C} \to \mathbb{C}$ be the map $s_1 + is_2 \mapsto \lambda s_1 + is_2/\lambda$. Note F is the time-1 map of the vector field V given by

$$\dot{s}_1 = (\log\lambda)s_1, \quad \dot{s}_2 = -(\log\lambda)s_2. \tag{3}$$

Let $0 < \rho_1 < \rho_0 < \min\{a_1, \ldots, a_\ell\} =: a^*$, and define r_0 and r_1 by $r_j = (2/p)\rho_j^{p/2}$ for $j = 0, 1$ and for each $p = p(k)$. Also let $\tilde{a} = (2/p)(a^*)^{p/2}$. Assume ρ_0, ρ_1 are chosen so that

$$D_{r_1} \subset F(D_{r_0}), \quad F(D_{r_1}) \cup F^{-1}(D_{r_0}) \subset D_{\tilde{a}}. \tag{4}$$

We also assume ρ_0 is chosen to be small enough so that the open neighborhood $\mathcal{U}_0 := \bigcup_{k=1}^m \phi_k^{-1}(D_{\rho_0})$ of the set S of singularities is disjoint from the open set $\bigcup_{k=m+1}^{\ell} \phi_k^{-1}(D_{a_k}) = \bigcup_{k=m+1}^{\ell} U_k$.

Let $\alpha \in (0,1)$ be a uniform constant. For each p-pronged singularity, define a "slow-down" function $\Psi_p = \Psi_{p,\alpha}$ on the interval $[0, \infty)$ so that:

1. $\Psi_p(u) = \left(\frac{p}{2}\right)^{2\alpha} u^\alpha$ for $u \le r_1^2$;
2. Ψ_p is C^∞ except at 0;
3. $\dot{\Psi}_p(u) \ge 0$ for $u > 0$;

4. $\Psi_p(u) = 1$ for $u \geq r_0^2$.

Consider the vector field $\hat{V}_p$ on $D_{r_0} \subset \mathbb{C}$ defined by

$$\dot{s}_1 = (\log\lambda) s_1 \Psi_p\left(s_1^2 + s_2^2\right) \quad \text{and} \quad \dot{s}_2 = -(\log\lambda) s_2 \Psi_p\left(s_1^2 + s_2^2\right). \tag{5}$$

Let G_p be the time-1 map of the vector field $\hat{V}_p$. Assume ρ_1 is chosen to be small enough so that $G_p = F$ on a neighborhood of the boundary of D_{r_0}.

Now let $\varphi : \mathbb{C} \to \mathbb{C}$ be given by

$$\varphi(z) = A\left(\int_0^{|z|^2} \frac{du}{\Psi_p(u)}\right)^{p/4} \frac{z}{|z|}, \tag{6}$$

where $A > 0$ is defined by

$$A = \left((1-\alpha)\left(\frac{p}{2}\right)^{2\alpha}\right)^{p/4}. \tag{7}$$

In particular, observe that for $0 < |z| < r_1$, we have $\Psi_p(u) = \left(\frac{p}{2}\right)^{2\alpha} u^\alpha$, and so for $|z| = r < r_1$:

$$\varphi(re^{i\theta}) = A\left(\int_0^{r^2} \left(\frac{p}{2}\right)^{-2\alpha} u^{-\alpha}\, du\right)^{p/4} e^{i\theta} = r^{p(1-\alpha)/2} e^{i\theta}. \tag{8}$$

Therefore near 0, denoting $\tilde{r}e^{i\tilde{\theta}} = \varphi(re^{i\theta})$, the coordinates (r, θ) and $(\tilde{r}, \tilde{\theta})$ are related by

$$\tilde{r} = r^{p(1-\alpha)/2}, \quad \text{and} \quad \tilde{\theta} = \theta.$$

For each singularity x_k, let $\tilde{a}_k = (2/p) a_k^{p/2}$, and define the coordinate change $\Phi_{kj} : \phi_k S_{kj}^s \to \{z : \mathrm{Re} z \geq 0\} \cap D_{\tilde{a}_k}$ by

$$\Phi_{kj}(z) = \Phi_{kj}(\rho e^{i\tau}) = (-1)^j \frac{2}{p} z^{p/2} = \frac{2}{p} \rho^{p/2} e^{i\tau\frac{p}{2} + ij\pi} = \tilde{r}e^{i\tilde{\theta}}. \tag{9}$$

Observe, therefore, that the coordinates (ρ, τ) and $(\tilde{r}, \tilde{\theta})$ are related by

$$\rho = \left(\frac{p}{2}\tilde{r}\right)^{2/p} \quad \text{and} \quad \tau = \frac{2}{p}\tilde{\theta} - \frac{2j\pi}{p}$$

Define $g : M \to M$ by $g(x) = f(x)$ for $x \notin \mathcal{U}_0$ and meanwhile for $1 \leq k \leq m$, $1 \leq j \leq p(k)$, define g on each sector $S_{kj}^s \cap \phi_k^{-1}(D_{\rho_0})$ by

$$g = \left(\varphi^{-1} \circ \Phi_{kj} \circ \phi_k\right)^{-1} \circ G_p \circ \left(\varphi^{-1} \circ \Phi_{kj} \circ \phi_k\right). \tag{10}$$

Note that $g = f$ in $\phi_k^{-1}(D_{a_k} \setminus D_{\rho_0})$, and therefore it follows from (4) that

$$\phi_k^{-1}(D_{\rho_1}) \subset g\left(\phi_k^{-1}(D_{\rho_0})\right), \quad g\left(\phi_k^{-1}(D_{r_1})\right) \cup g^{-1}\left(\phi_k^{-1}(D_{r_0})\right) \subset \phi_k^{-1}(D_{\tilde{a}}). \tag{11}$$

Remark 4.1 In the original smooth pseudo-Anosov realization constructed in [2], the exponent they chose is $\alpha = (p-2)/p$, in which case one can compute $\varphi = \mathrm{id}$. We emphasize that here we allow α to take on *any value* in (0,1). This is necessary because proving polynomial decay of correlations requires $\alpha < 1/4$, which does not work if $\alpha = (p-2)/p$ and the number of prongs at a singularity is $p \geq 3$. Further proving the central limit theorem requires $\alpha < 1/6$, so we assume $\alpha < 1/6$ throughout. See Remark 9.3.

4.2 Smoothness and area invariance of g

We now show that g is a $C^{2+\beta}$ diffeomorphism on M and preserves a smooth invariant measure. Let $x_k \in M$ be a singularity of g. Consider the vector field V given by (3) defined on $D_{r_1} = (\varphi^{-1} \circ \Phi_{kj})(D_{\rho_1})$, and let $\Omega = ds_1 \wedge ds_2 = r dr \wedge d\theta$ be the Lebesgue area form. Observe that V is Hamiltonian with respect to Ω, with Hamiltonian function $H(s_1, s_2) = s_1 s_2 \log \lambda$. Define the area form $\hat{\Omega}_p$ by

$$(\hat{\Omega}_p)_{(s_1,s_2)} = \frac{ds_1 \wedge ds_2}{\Psi_p(s_1^2 + s_2^2)} = \frac{r dr \wedge d\theta}{\Psi_p(r^2)}.$$

Note the vector field $\hat{V}_p$ defined by (5) is Hamiltonian with respect to $\hat{\Omega}_p$, with Hamiltonian function H. Finally let V_p be the (continuous) vector field on $D_{a_k} \subset \mathbb{C}$ given by $(\Phi_{kj}^{-1} \circ \varphi)_* \hat{V}_p$, and let $\Omega_p = (\varphi^{-1} \circ \Phi_{kj})^* \hat{\Omega}_p$. Note V_p is Hamiltonian with respect to Ω_p, with Hamiltonian function $H_p := H \circ \varphi^{-1} \circ \Phi_{kj}$.

Lemma 4.2 *Near the origin, Ω_p is a constant times Lebesgue area in D_{a_k}.*

Proof Note that for $\rho > 0$ sufficiently small, the function $\left(\varphi^{-1} \circ \Phi_{kj}\right)(\rho e^{i\tau}) = re^{i\theta}$ satisfies

$$\begin{aligned} re^{i\theta} &= \left(\varphi^{-1} \circ \Phi_{kj}\right)(\rho e^{i\tau}) \\ &= \varphi^{-1}\left(\frac{2}{p}\rho^{p/2} e^{i\tau\frac{p}{2} + ij\pi}\right) \\ &= \left(\frac{2}{p}\right)^{2/p(1-\alpha)} \rho^{1/(1-\alpha)} e^{i\tau\frac{p}{2} + ij\pi}, \end{aligned} \tag{12}$$

and so the coordinates (r, θ) and (ρ, τ) are related by

$$r = \left(\frac{2}{p}\right)^{2/p(1-\alpha)} \rho^{1/(1-\alpha)}, \quad \theta = \frac{p}{2}\tau + j\pi.$$

It follows that:

$$dr = \frac{1}{1-\alpha}\left(\frac{2}{p}\right)^{2/p(1-\alpha)} \rho^{\alpha/(1-\alpha)} d\rho \quad \text{and} \quad d\theta = \frac{p}{2} d\tau.$$

So, since in polar coordinates we can write $\hat{\Omega}_p = \frac{1}{\Psi_p(r^2)} r dr \wedge d\theta$, for $\rho e^{i\tau}$ sufficiently near 0, we have:

$$\begin{aligned}
\Omega_p &= \left(\varphi^{-1} \circ \Phi_{kj}\right)^* \hat{\Omega}_p \\
&= \left(\varphi^{-1} \circ \Phi_{kj}\right)^* \left(\frac{r dr \wedge d\theta}{\Psi_p(r^2)}\right) \\
&= \frac{1}{\Psi_p\left(\left(\frac{2}{p}\right)^{4/p(1-\alpha)} \rho^{2/(1-\alpha)}\right)} \times \left(\left(\frac{2}{p}\right)^{2/p(1-\alpha)} \rho^{1/(1-\alpha)}\right) \\
&\quad \times \left(\frac{1}{1-\alpha}\left(\frac{2}{p}\right)^{2/p(1-\alpha)} \rho^{\alpha/(1-\alpha)} d\rho\right) \wedge \left(\frac{p}{2} d\tau\right) \\
&= \left(\left(\frac{p}{2}\right)^{-2\alpha}\left(\frac{p}{2}\right)^{\frac{4\alpha}{p(1-\alpha)}} \rho^{-\frac{2\alpha}{1-\alpha}}\right) \times \left(\left(\frac{p}{2}\right)^{-\frac{2}{p(1-\alpha)}} \rho^{\frac{1}{1-\alpha}}\right) \\
&\quad \times \left(\frac{1}{1-\alpha}\left(\frac{p}{2}\right)^{1-\frac{2}{p(1-\alpha)}} \rho^{\frac{\alpha}{1-\alpha}}\right) d\rho \wedge d\tau \\
&= \frac{1}{1-\alpha}\left(\frac{p}{2}\right)^{1-\frac{4}{p}-2\alpha} \rho d\rho \wedge d\tau.
\end{aligned} \tag{13}$$

Since $\rho d\rho \wedge d\tau$ is the Lebesgue area in D_{a_k}, we've proven the lemma. □

Remark 4.3 In the original smooth pseudo-Anosov realization constructed in [2], the exponent they chose is $\alpha = (p-2)/p$, in which case one can compute that the constant in front of $\rho d\rho \wedge d\tau$ in the final equality of (13) is 1, and the area is precisely Lebesgue area.

Recall $V_p = (\Phi_{kj}^{-1} \circ \varphi)_* \hat{V}_p$, where $\hat{V}_p$ is given by (5). Note $\text{div}_\Omega V = 0$, and it follows that $\text{div}_{\hat{\Omega}_p} \hat{V}_p = 0$, and so $\text{div}_{\Omega_p} V_p = 0$ in a neighborhood of each singularity. Since g is the time-1 map of V_p on M, one can use a partition of unity on $(U_k, \phi_k)_{1 \le k \le \ell}$ and the coordinate representation of g in each chart to prove:

Proposition 4.4 *The map $g : M \to M$ preserves a smooth invariant measure μ_1 that is equivalent to the Riemannian area on M.*

We next show g is $C^{2+\beta}$. To do this, we need the following technical result:

Lemma 4.5 *Suppose $f(t_1, t_2) = C|(t_1, t_2)|^\delta Q(t_1, t_2)$, where $\delta > 0$, $|(t_1, t_2)| = \sqrt{t_1^2 + t_2^2}$ and $Q : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial whose terms are all of order p. That is,*

$$Q(t_1, t_2) = \sum_{j=0}^{p} A_j t_1^j t_2^{p-j}, \quad A_j \in \mathbb{R}.$$

Then, for $j = 1, 2$,

$$\frac{\partial f}{\partial t_j} = C|(t_1, t_2)|^{\delta - 2} Q_1(t_1, t_2), \tag{14}$$

where Q_1 is a polynomial whose terms are all of order $p+1$. In particular, inductively, it follows that for every $k \geq 1$ and every $0 \leq \ell \leq k$,

$$\frac{\partial^k f}{\partial t_1^\ell \partial t_2^{k-\ell}} = C|(t_1,t_2)|^{\delta-2k} Q_k(t_1,t_2) \tag{15}$$

where Q_k is a poynomial whose terms are all of degree $p+k$.

Proof If $Q(t_1,t_2)$ has monomial terms all of degree p, then $Q_{t_j}(t_1,t_2)$ has terms all of degree $p-1$. Meanwhile,

$$\frac{\partial}{\partial t_j}|(t_1,t_2)|^\delta = \frac{\partial}{\partial t_j}\left(t_1^2+t_2^2\right)^{\delta/2} = \delta t_j\left(t_1^2+t_2^2\right)^{(\delta-2)/2} = \delta t_j|(t_1,t_2)|^{\delta-2}$$

If $f(t_1,t_2) = C|(t_1,t_2)|^\delta Q(t_1,t_2)$, it follows that

$$\begin{aligned}\frac{\partial f}{\partial t_j} &= C\left(t_j|(t_1,t_2)|^{\delta-2}Q(t_1,t_2) + |(t_1,t_2)|^\delta Q_{t_j}(t_1,t_2)\right)\\ &= C|(t_1,t_2)|^{\delta-2}\left(t_jQ(t_1,t_2) + (t_1^2+t_2^2)Q_{t_j}(t_1,t_2)\right).\end{aligned}$$

(14) now follows with $Q_1(t_1,t_2) = t_jQ(t_1,t_2) + (t_1^2+t_2^2)Q_{t_j}(t_1,t_2)$, and (15) follows by induction. □

Proposition 4.6 *$H_p = H \circ \varphi^{-1} \circ \Phi_{kj}$ is at least $C^{2+\beta}$, where $\beta = \frac{2}{1-\alpha} - \left\lfloor \frac{2}{1-\alpha} \right\rfloor$.*

Proof Note $H(s_1,s_2) = s_1s_2\log\lambda$ in polar coordinates is

$$H(re^{i\theta}) = (\log\lambda)r^2\cos\theta\sin\theta = \frac{1}{2}(\log\lambda)r^2\sin(2\theta).$$

It follows from (12) that for $\rho e^{i\tau} = t_1 + it_2$, we have:

$$\begin{aligned}H_p(\rho e^{i\tau}) &= \frac{1}{2}(\log\lambda)\left(\frac{2}{p}\right)^{4/p(1-\alpha)}\rho^{2/(1-\alpha)}\sin(\tau p)\\ &= \frac{1}{2}(\log\lambda)\left(\frac{2}{p}\right)^{4/p(1-\alpha)}\rho^{\frac{2-p(1-\alpha)}{1-\alpha}}\rho^p\sin(\tau p)\\ &= \frac{1}{2}(\log\lambda)\left(\frac{2}{p}\right)^{4/p(1-\alpha)}|t_1+it_2|^{\frac{2}{1-\alpha}-p}\mathrm{Im}(z^p).\end{aligned} \tag{16}$$

Since $\mathrm{Im}(z^p)$ is a polynomial in t_1 and t_2 whose monomial terms are all of order p, Lemma 4.5 gives us that for $k \geq 1$, $0 \leq \ell \leq k$,

$$\frac{\partial^k H_p}{\partial t_1^\ell \partial t_2^{k-\ell}} = \frac{1}{2}(\log\lambda)\left(\frac{2}{p}\right)^{4/p(1-\alpha)}|t_1+it_2|^{\frac{2}{1-\alpha}-p-2k}Q_k(t_1,t_2), \tag{17}$$

where Q_k is a polynomial whose monomial terms are of degree $p+k$. In other words,

$$Q_k(t_1,t_2) = Q_k(\rho e^{i\tau}) = \rho^{p+k}h(\tau),$$

where $h : [0,2\pi] \to \mathbb{R}$ is a continuous and bounded function. It follows from (17) that

$$\partial_{t_1}^\ell \partial_{t_2}^{k-\ell} H_p := \frac{\partial^k H_p}{\partial t_1^\ell \partial t_2^{k-\ell}}(\rho e^{i\tau}) = B\rho^{\frac{2}{1-\alpha}-p-2k+(p+k)}h(\tau) = B\rho^{\frac{2}{1-\alpha}-k}h(\tau), \tag{18}$$

where $B > 0$ is a constant. This function is continuous on $\mathbb{C}$ as long as $k < \frac{2}{1-\alpha}$. Note $\frac{2}{1-\alpha} > 2$ since $0 < \alpha < 1$. For $k = \left\lfloor \frac{2}{1-\alpha} \right\rfloor$, it follows that H_p is $C^{k+\beta}$, $\beta = \frac{2}{1-\alpha} - \left\lfloor \frac{2}{1-\alpha} \right\rfloor$. □

Since the vector field V_p is Hamiltonian with respect to Lebesgue area with Hamiltonian function H_p, it follows that V_p is $C^{2+\beta}$, and thus the map $g : M \to M$ is $C^{2+\beta}$ (note g is C^∞ away from the singularities).

4.3 Other topological properties

The smooth realization g of a pseudo-Anosov homeomorphism f is adapted from a smooth realization of pseudo-Anosov homeomorphisms first described in [2]. In this construction, the slow-down exponent α in the definition of Ψ_p is taken to be $\alpha = (p-2)/p$. It follows that the homeomorphism $\varphi : \mathbb{C} \to \mathbb{C}$ is the identity and the Hamiltonian function H_p of V_p is a constant times $\text{Im}(z^p)$, i.e., a polynomial (see (16)), and hence V_p is analytic. Therefore the smooth pseudo-Anosov model in [2] is analytic, not just $C^{2+\varepsilon}$. However, using similar arguments to Section 6.4 of [28], $C^{2+\varepsilon}$ is sufficient regularity to prove the following:

Proposition 4.7 *The smooth pseudo-Anosov realization $g : M \to M$ defined by* (10) *has the following properties:*

1. *g is topologically conjugate to the linear pseudo-Anosov homeomorphism f, via a continuous (but not C^1) conjugacy h isotopic to the identity.*
2. *For any $\varepsilon > 0$, one can choose α, ρ_0, and ρ_1 in the construction of g so that $\|f - g\|_{C^0} < \varepsilon$.*
3. *The map g admits two invariant distributions $x \mapsto E^u(x), E^s(x)$, which are continuous on M except at the singularities. At μ_1-a.e. $x \in M$ (where μ_1 is the measure in Proposition 4.4), g admits two nonzero Lyapunov exponents: one negative exponent in the direction of $E^s(x)$, and one positive exponent in the direction of $E^u(x)$.*
4. *The map g admits two invariant foliations with singularities of M, which are the images under the conjugating homeomorphism h of the foliations with singularities $\mathcal{F}^s$ and $\mathcal{F}^u$ associated to the pseudo-Ansoov homeomorphism f.*
5. *The map g admits a finite Markov partition, given by the image of the Markov partition of f under the conjugating homeomorphism h.*

Finally, in the case when the slow-down exponent is $\alpha = (p-2)/p$, it is shown in [7] that the geometric t-potentials $\varphi_t(x) = -t \log \left| Dg|_{E^u(x)} \right|$ admit unique equilibrium states for $t_0 < t < 1$, $t_0 < 0$, which includes a unique measure of maximal entropy. Furthermore, these equilibrium states have exponential decay of correlations and the Central Limit Theorem with respec to Hölder-continuous potentials. Using identical techniques in [5] and [7], as well as results from [30] and [31], this result extends verbatim to pseudo-Anosov smooth realizations with arbitrary slow-down exponents $0 < \alpha < 1$:

Proposition 4.8 *The following hold for the pseudo-Anosov smooth realization g:*

1. *Given any $t_0 < 0$, we may take $\rho_0 > 0$ in the construction of g so that for any $t \in (t_0, 1)$, there is a unique equilibrium measure μ_t associated to φ_t. This equilibrium measure has nonzero Lyapunov exponents almost everywhere, exponential decay of correlations and satisfies the Central Limit Theorem with respect to a class of functions containing all Hölder continuous functions on M, and is Bernoulli. Additionally, the pressure function $t \mapsto P_g(\varphi_t)$ is real analytic in the open interval $(t_0, 1)$.*
2. *For $t = 1$, there are two classes of equilibrium measures associated to φ_1: convex combinations of Dirac measures concentrated at the singularities, and a unique invariant SRB measure μ.*
3. *For $t > 1$, the equilibrium measures associated to φ_t are precisely the convex combinations of Dirac measures concentrated at the singularities.*

5 Pseudo-Anosov Diffeomorphisms are Young Diffeomorphisms

5.1 Young diffeomorphisms

The proof of Theorem 3.1 relies on recent results on the thermodynamics of Young diffeomorphisms. In this section, we define Young diffeomorphisms and describe some of their thermodynamic properties. The following description of Young diffeomorphisms is discussed in Section 4 of [5] and Section 6 of [7], and is printed here for the reader's convenience.

Given a $C^{1+\alpha}$ diffeomorphism f on a compact Riemannian manifold M, we call an embedded C^1 disc $\gamma \subset M$ an *unstable disc* (resp. *stable disc*) if for all $x, y \in \gamma$, we have $d(f^{-n}(x), f^{-n}(y)) \to 0$ (resp. $d(f^n(x), f^n(y)) \to 0$) as $n \to +\infty$. A collection of embedded C^1 discs $\Gamma = \{\gamma_i\}_{i \in \mathcal{I}}$ is a *continuous family of unstable discs* if there is a Borel subset $K^s \subset M$ and a homeomorphism $\Phi : K^s \times D^u \to \bigcup_i \gamma_i$, where $D^u \subset \mathbb{R}^d$ is the closed unit disc for some $d < \dim M$, satisfying:

- The assignment $x \mapsto \Phi|_{\{x\} \times D^u}$ is a continuous map from K^s to the space of C^1 embeddings $D^u \hookrightarrow M$, and this assignment can be extended to the closure $\overline{K^s}$;
- For every $x \in K^s$, $\gamma = \Phi(\{x\} \times D^u)$ is an unstable disc in Γ.

Thus the index set $\mathcal{I}$ may be taken to be $K^s \times \{0\} \subset K^s \times D^u$. We define *continuous families of stable discs* analogously.

A subset $\Lambda \subset M$ has *hyperbolic product structure* if there is a continuous family $\Gamma^u = \{\gamma_i^u\}_{i \in \mathcal{I}}$ of unstable discs and a continuous family $\Gamma^s = \{\gamma_j^s\}_{j \in \mathcal{J}}$ of stable discs such that

- $\dim \gamma_i^u + \dim \gamma_j^s = \dim M$ for all i, j;
- the unstable discs are transversal to the stable discs, with an angle uniformly bounded away from 0;
- each unstable disc intersects each stable disc in exactly one point;
- $\Lambda = \left(\bigcup_i \gamma_i^u\right) \cap \left(\bigcup_j \gamma_j^s\right)$.

A subset $\Lambda_0 \subset \Lambda$ with hyperbolic product structure is an *s-subset* if the continuous family of unstable discs defining Λ_0 is the same as the continuous family of unstable discs for Λ, and the continuous family of stable discs defining Λ_0 is a subfamily Γ_0^s of the continuous family of stable discs defining Γ_0. In other words, if $\Lambda_0 \subset \Lambda$ has hyperbolic product structure generated by the families of stable and unstable discs given by Γ_0^s and Γ_0^u, then Λ_0 is an s-subset if $\Gamma_0^s \subseteq \Gamma^s$ and $\Gamma_0^u = \Gamma^u$. A *u-subset* is defined analogously.

Definition 5.1 A $C^{1+\alpha}$ diffeomorphism $f : M \to M$, with M a compact Riemannian manifold, is a *Young diffeomorphism* if the following conditions are satisfied:

1. There exists $\Lambda \subset M$ (called the *base*) with hyperbolic product structure, a countable collection of continuous subfamilies $\Gamma_i^s \subset \Gamma^s$ of stable discs, and positive integers τ_i, $i \in \mathbb{N}$, such that the s-subsets

$$\Lambda_i^s := \bigcup_{\gamma \in \Gamma_i^s} \left(\gamma \cap \Lambda\right) \subset \Lambda$$

are pairwise disjoint and satisfy:

(a) *invariance*: for $x \in \Lambda_i^s$,

$$f^{\tau_i}(\gamma^s(x)) \subset \gamma^s(f^{\tau_i}(x)), \quad \text{and} \quad f^{\tau_i}(\gamma^u(x)) \supset \gamma^u(f^{\tau_i}(x)),$$

where $\gamma^{u,s}(x)$ denotes the (un)stable disc containing x; and,

(b) *Markov property*: $\Lambda_i^u := f^{\tau_i}(\Lambda_i^s)$ is a u-subset of Λ such that for $x \in \Lambda_i^s$,

$$f^{-\tau_i}(\gamma^s(f^{\tau_i}(x)) \cap \Lambda_i^u) = \gamma^s(x) \cap \Lambda, \quad \text{and} \quad f^{\tau_i}(\gamma^u(x) \cap \Lambda_i^s) = \gamma^u(f^{\tau_i}(x)) \cap \Lambda.$$

2. For $\gamma^u \in \Gamma^u$, we have

$$\mu_{\gamma^u}(\gamma^u \cap \Lambda) > 0, \quad \text{and} \quad \mu_{\gamma^u}\Big(\mathrm{cl}\big((\Lambda \setminus \textstyle\bigcup_i \Lambda_i^s) \cap \gamma^u \big)\Big) = 0,$$

where μ_{γ^u} is the induced Riemannian leaf volume on γ^u and $\mathrm{cl}(A)$ denotes the closure of A in M for $A \subseteq M$.

3. There is $a \in (0,1)$ so that for any $i \in \mathbb{N}$, we have:

(a) For $x \in \Lambda_i^s$ and $y \in \gamma^s(x)$,

$$d(F(x), F(y)) \le a d(x,y);$$

(b) For $x \in \Lambda_i^s$ and $y \in \gamma^u(x) \cap \Lambda_i^s$,

$$d(x,y) \le a d(F(x), F(y)),$$

where $F : \bigcup_i \Lambda_i^s \to \Lambda$ is the *induced map* defined by

$$F|_{\Lambda_i^s} := f^{\tau_i}|_{\Lambda_i^s}.$$

4. Denote $J^u F(x) = \det \left| DF|_{E^u(x)} \right|$. There exist $c > 0$ and $\kappa \in (0, 1)$ such that:
   (a) For all $n \geq 0$, $x \in F^{-n} \left(\bigcup_i \Lambda_i^s \right)$ and $y \in \gamma^s(x)$, we have

   $$\left| \log \frac{J^u F(F^n(x))}{J^u F(F^n(y))} \right| \leq c\kappa^n;$$

   (b) For any $i_0, \ldots, i_n \in \mathbb{N}$ with $F^k(x), F^k(y) \in \Lambda_{i_k}^s$ for $0 \leq k \leq n$ and $y \in \gamma^u(x)$, we have

   $$\left| \log \frac{J^u F(F^{n-k}(x))}{J^u F(F^{n-k}(y))} \right| \leq c\kappa^k.$$

5. There is some $\gamma^u \in \Gamma^u$ such that

$$\sum_{i=1}^{\infty} \tau_i \mu_{\gamma^u} \left(\Lambda_i^s \right) < \infty.$$

5.2 Realizing *g* as a Young diffeomorphism

In Section 7 of [7], it is shown that the smooth nonuniformly hyperbolic pseudo-Anosov diffeomorphism $g : M \to M$ is a Young diffeomorphism. We briefly outline the argument here.

The first step is to show that the (uniformly hyperbolic) pseudo-Anosov homeomorphism $f : M \to M$ admits a subset $\tilde{\Lambda} \subset M$ for which Conditions (Y1) - (Y5) are satisfied. To construct $\tilde{\Lambda}$, we use a finite Markov partition $\tilde{\mathcal{P}}$ for the pseudo-Anosov homeomorphism f (Proposition 2.7). Note that if $\tilde{R} \in \tilde{\mathcal{P}}$ is a Markov rectangle, then no singularity of f may lie inside the interior of $\tilde{R}$ (intuitively this is because f does not admit local hyperbolic product structure at the singularities). Thus, we may take our Markov partition $\tilde{\mathcal{P}}$ of f to be so that the singularities lie on the vertices of the rectangles.

Recall that $S = \{x_1, \ldots, x_m\}$ denotes the set of singularities, each of which has $p(x_k) = p(k)$ prongs $(1 \leq k \leq m)$, i.e., the stable foliation $\tilde{\mathcal{F}}^s$ and the unstable foliation $\tilde{\mathcal{F}}^u$ of f each have p_k prongs at x_k. Since each singularity x_k is the vertex of a Markov rectangle, there are $2p(k)$ Markov rectangles with x_k as a vertex; we denote these rectangles by $\tilde{R}_{k,l}$ for $1 \leq k \leq m$, $1 \leq l \leq 2p(k)$. By choosing the diameter of the partition elements to be sufficiently small, we may assume that $\tilde{R}_{k_1,l_1} \cap \tilde{R}_{k_2,l_2} = \emptyset$ whenever $k_1 \neq k_2$.

Let $\tilde{R} \in \tilde{\mathcal{P}}$ be a partition element that does not intersect the set $\mathcal{U}_0$ defined in the slow-down procedure for the map g. For $x \in \tilde{R}$, let $\tilde{\gamma}^s(x)$ and $\tilde{\gamma}^u(x)$ respectively be the connected components of the stable and unstable leaves through x intersecting $\tilde{R}$. We call these the *full-length* stable and unstable curves through x.

Let $\tilde{\tau}(x)$ be the first return time of x to int$\tilde{R}$ under f for $x \in \tilde{R}$. For all x with $\tilde{\tau}(x) < \infty$, define the set:

$$\tilde{\Lambda}^s(x) = \bigcup_{y \in \tilde{U}^u(x) \setminus \tilde{A}^u(x)} \tilde{\gamma}^s(y),$$

where $\tilde{U}^u(x) \subset \tilde{\gamma}^u(x)$ is an interval containing x and open in the induced topology of $\tilde{\gamma}^u(x)$, and $\tilde{A}^u(x) \subset \tilde{U}^u(x)$ is the set of points either lying on the boundary of the Markov partition or never return to the set $\tilde{P}$. Observe that $\tilde{A}^u(x)$ has one-dimensional Lebesgue measure equal to 0. One can choose the intervals $\tilde{U}^u(x)$ so that

1. for any $y \in \tilde{\Lambda}^s(x)$, we have $\tilde{\tau}(y) = \tilde{\tau}(x)$; and
2. for any $y \in \tilde{R}$ with $\tilde{\tau}(y) < \infty$, there is an $x \in \tilde{R}$ for which $y \in \tilde{\Lambda}^s(x)$ and $\tilde{\tau}(y) = \tilde{\tau}(x)$.

Moreover, the image of $\tilde{\Lambda}^s(x)$ under $f^{\tilde{\tau}(x)}$ is a u-subset containing $f^{\tilde{\tau}(x)}(x)$. Note that conditions (1) and (2) above ensure that for $x, y \in \tilde{R}$ with finite return times, the sets $\tilde{\Lambda}^s(x)$ and $\tilde{\Lambda}^s(y)$ either coincide or are disjoint. Thus we have a countable collection of disjoint sets $\tilde{\Lambda}^s_i$ and numbers $\tilde{\tau}_i$ that give a representation of the pseudo-Anosov homeomorphism f as a Young diffeomorphism with tower base

$$\tilde{\Lambda} = \bigcup_{i \geq 1} \tilde{\Lambda}^s_i.$$

The sets $\tilde{\Lambda}^s_i$ form the s-sets, $\tilde{\Lambda}^u_i = f^{\tilde{\tau}_i}(\tilde{\Lambda}^s_i)$ form the u-sets, and the numbers $\tilde{\tau}_i$ form the inducing times. See Theorem 7.1 in [7] for details.

Let $h : M \to M$ denote the conjugacy map between f and g, so that $g = h \circ f \circ h^{-1}$. Applying h to the Markov partition $\tilde{\mathcal{P}}$, one obtains a Markov partition $\mathcal{P} = h(\tilde{\mathcal{P}})$ of the pseudo-Anosov diffeomorphism g. By continuity of h, one can construct a Markov partition of g in this way of arbitrarily small diameter. Let $R = h(\tilde{R})$, $\Lambda = h(\tilde{\Lambda})$. Observe that Λ has local hyperbolic product structure given by the full-length stable leaves $\gamma^s(x) = h(\tilde{\gamma}^s(h^{-1}(x)))$ and the full-length unstable leaves $\gamma^u(x) = h(\tilde{\gamma}^u(h^{-1}(x)))$. Accordingly, it is shown in [7] that g is represented as a Young diffeomorphism with inducing times $\tau_i = \tilde{\tau}_i$, s-sets $\Lambda^s_i = h(\tilde{\Lambda}^s_i)$, and u-sets $\Lambda^u_i = h(\tilde{\Lambda}^u_i) = g^{\tau_i}(\Lambda^s_i)$. Similarly to the homeomorphism f, the inducing times τ_i are first-return times to Λ for points $x \in \Lambda^s_i$ under the g. Furthermore, note that if $x \in \Lambda^s_i$, the stable subset Λ^s_i satisfies

$$\Lambda^s_i = \Lambda^s(x) = \bigcup_{y \in U^u(x) \setminus A^u(x)} \gamma^s(y),$$

where $U^u(x) = h(\tilde{U}^u(x)) \subset \gamma^u(x)$ is an interval containing x and open in the induced topology of $\gamma^u(x)$, and $A^u(x) = h(\tilde{A}^u(x)) \subset U^u(x)$ is the set of points that either lie on the boundary of the Markov partition $\mathcal{P}$ or never return to R. Observe that $A^u(x)$ has one-dimensional Lebesgue measure equal to 0 in $\gamma^u(x)$.

Proposition 5.2 *Given $Q > 0$, one can choose a Markov partition $\mathcal{P}$ for g and the number r_0 in the construction of g so that*

1. *$g^j(x) \notin \mathcal{U}_0$ for any $0 \leq j \leq Q$ and for any point $x \in M$ for which either $x \in \Lambda$ or $x \notin \mathcal{U}_0$, while $g^{-1}(x) \in \mathcal{U}_0$; and,*
2. *if $R_{k,l} = h(\tilde{R}_{k,l})$, with $R_{k,l}$ a Markov rectangle with the singularity x_k as a vertex ($1 \leq k \leq m$, $1 \leq l \leq 2p(k)$), then*

$$\mathcal{U}_0 \subset \operatorname{int} \bigcup_{k=1}^{m} \bigcup_{l=1}^{2p(k)} R_{k,l}. \tag{19}$$

To prove this proposition, simply note that it holds for the pseudo-Anosov homeomorphism f. Applying the conjugacy h yields the result.

Proposition 5.3 ([7]) *There is a $Q > 0$ such that the collection of s-sets Λ_i^s satisfies Conditions (Y1) - (Y5), thus representing $g : M \to M$ as a Young diffeomorphism.*

6 Behavior near singularities

In this section, we consider specifically the behavior of trajectories of the system of differential equations given by (5) in (s_1, s_2)-coordinates. The computations in this section pertain specifically to this system of ODEs, and have no *a priori* relation to the manifold M, the pseudo-Anosov map f, or its smooth realization g.

Remark 6.1 Many of the results on the behavior of this system of ODEs that we cite in this section are proven in [4] and [5]. In [4, 5], they use a slow-down function $\psi : [0, 1] \to \mathbb{R}$ for which there is an $0 < r_0 < 1$ such that for $u < (r_0/2)^2$,

$$\psi(u) = \left(\frac{u}{r_0}\right)^{\alpha}.$$

On the other hand, the slow-down function $\Psi_p : [0, 1] \to \mathbb{R}$ that we use has constants $0 < r_1 < r_0 < 1$ for which for $u < r_1^2$, we have

$$\Psi_p(u) = \left(\frac{p}{2}\right)^{2\alpha} u^{\alpha}.$$

In other words, the coefficient $r_0^{-\alpha}$ has been replaced with the coefficient $(p/2)^{2\alpha}$. For this reason, up to a constant multiple, the system of differential equations (5) is the same as the respective system of differential equations in [4, 5]. Accordingly, the results we cite here are proven in [4, 5], up to a multiplicative constant. Several proofs are omitted in this section in the intersest of brevity, but references are given for the respective results in [4, 5].

Our next several lemmas concern the trajectories of solutions to equation (5). Let $s(t) = (s_1(t), s_2(t))$ be a solution to (5). Assume $s(t)$ is defined in the maximal interval $[0, T]$, for which $s(0), s(T) \in \partial D_{r_1}$ and $s(t) \in D_{r_1}$ for $0 < t < T$. Further let $T_1 = T/2$. Note $s_1(t) \leq s_2(t)$ for $0 \leq t \leq T_1$ and $s_1(t) \geq s_2(t)$ for $T_1 \leq t \leq T$. We collect lower and upper bounds on the functions $s_1(t)$ and $s_2(t)$.

Lemma 6.2 *Given a solution $s(t)$ to (5), and T and T_1 defined above, we have the following estimates:*

1. $|s_2(t)| \geq |s_2(a)| \left(1 + 2^\alpha C_0 s_2^{2\alpha}(a)(t-a)\right)^{-1/2\alpha}, \quad 0 \leq a \leq t \leq T_1$;
2. $|s_2(t)| \leq |s_2(a)| \left(1 + C_0 s_2^{2\alpha}(a)(t-a)\right)^{-1/2\alpha}, \quad 0 \leq a \leq t \leq T$;
3. $|s_1(t)| \geq |s_1(b)| \left(1 + 2^\alpha C_0 s_1^{2\alpha}(b)(b-t)\right)^{-1/2\alpha}, \quad T_1 \leq t \leq b \leq T$;
4. $|s_1(t)| \leq |s_1(b)| \left(1 + C_0 s_1^{2\alpha}(b)(b-t)\right)^{-1/2\alpha}, \quad 0 \leq t \leq b \leq T$;

where $C_0 = 2\alpha \log \lambda (p/2)^{2\alpha}$.

Proof Assume $s_1(t), s_2(t) > 0$ for all $0 \leq t \leq T$. Equation (5) with $\Psi_p(u) = (p/2)^{2\alpha} u^\alpha$ for $0 \leq \alpha \leq r_1^2$ gives us, for $0 \leq t \leq T$ and $i = 1, 2$,

$$\frac{ds_i}{dt} = (-1)^{i+1} \log \lambda \left(\frac{p}{2}\right)^{2\alpha} s_i \left(s_1^2 + s_2^2\right)^\alpha. \tag{20}$$

Since $s_i^2 \leq s_1^2 + s_2^2$, we have

$$\frac{ds_1}{dt} \geq \log \lambda \left(\frac{p}{2}\right)^{2\alpha} s_1^{2\alpha+1} \quad \text{and} \quad \frac{ds_2}{dt} \leq -\log \lambda \left(\frac{p}{2}\right)^{2\alpha} s_2^{2\alpha+1}.$$

In particular, this implies

$$s_1(t)^{-2\alpha-1} \frac{ds_1(t)}{dt} \geq \log \lambda \left(\frac{p}{2}\right)^{2\alpha} \quad \text{and} \quad s_2(t)^{-2\alpha-1} \frac{ds_2(t)}{dt} \leq -\log \lambda \left(\frac{p}{2}\right)^{2\alpha}. \tag{21}$$

Integrating the inequalities in (21) over the interval $[a, b] \subset [0, T]$ yields

$$-\frac{1}{2\alpha} \left(s_1(b)^{-2\alpha} - s_1(a)^{-2\alpha}\right) \geq \log \lambda \left(\frac{p}{2}\right)^{2\alpha} (b-a)$$

and

$$-\frac{1}{2\alpha} \left(s_2(b)^{-2\alpha} - s_2(a)^{-2\alpha}\right) \leq -\log \lambda \left(\frac{p}{2}\right)^{2\alpha} (b-a),$$

or in other words,

$$s_1(b)^{-2\alpha} - s_1(a)^{-2\alpha} \leq -C_0(b-a) \quad \text{and} \quad s_2(b)^{-2\alpha} - s_2(\alpha)^{-2\alpha} \geq C_0(b-a).$$

Inequalities (b) and (d) all follow by setting $t = a$ or $t = b$.

Now, for $0 \leq t \leq T_1$, we have $s_2(t) \geq s_1(t)$, and for $T_1 \leq t \leq T$, we have $s_1(t) \geq s_2(t)$. So,

$$s_1^2 + s_2^2 \leq 2s_2^2 \quad \text{for } 0 \leq t \leq T_1$$

and

$$s_1^2 + s_2^2 \leq 2s_1^2 \quad \text{for } t_1 \leq t \leq T.$$

It follows from (20) that

$$\frac{ds_2}{dt} \geq -2^\alpha \log \lambda \left(\frac{p}{2}\right)^{2\alpha} s_2^{2\alpha+1} \quad \text{for } 0 \leq t \leq T_1$$

and

$$\frac{ds_1}{dt} \leq 2^\alpha \log \lambda \left(\frac{p}{2}\right)^{2\alpha} s_1^{2\alpha+1} \quad \text{for } T_1 \leq t \leq T.$$

Inequalities (a) and (c) can now be proven in a similar way to inequalities (b) and (d). □

Consider another solution $\tilde{s}(t)$ of (5) for which $s(0)$ and $\tilde{s}(0)$ lie in the same quadrant. Set $\Delta s(t) = \tilde{s}(t) - s(t)$ and $\Delta s_j(t) = \tilde{s}_j(t) - s_j(t)$, $j = 1, 2$.

Lemma 6.3 (([5], Lemma 5.3 and errata)) *Suppose* $s_1(t) \neq 0 \neq s_2(t)$ *for* $t \in [0,T]$ *and that* $|\tilde{s}_2(t)| > |s_2(t)|$ *for* $t \in [0,T]$. *Suppose further that* $0 < \mu < 1$ *satisfies*

1. $\Delta s_2(t) > 0$ *and* $|\Delta s_1(t)| \leq \mu |\Delta s_2(t)|$ *for* $t \in [0,T]$;
2. $\left|\frac{\Delta s_2(0)}{s_2(0)}\right| \leq \frac{1-\mu}{72}$.

Then,

$$\Delta s_2(t) \leq \left|\frac{\Delta s_2(0)}{s_2(0)}\right| |s_2(t)| \left(1 + 2^\alpha C_0 |s_2(0)|^{2\alpha} t\right)^{-\beta}, \qquad 0 \leq t \leq T_1,$$

$$\Delta s_2(t) \leq \left|\frac{\Delta s_2(T_1)}{s_1(T_1)}\right| |s_1(t)| \left(\frac{1 + 2^\alpha C_0 |s_1(b)|^{2\alpha}(b-t)}{1 + 2^\alpha C_0 |s_1(b)|^{2\alpha}(b-T_1)}\right)^{\beta}, \qquad T_1 \leq t \leq b \leq T,$$

where $\beta = \frac{1-\mu}{2^{\alpha+2}}$, *and* C_0 *is the constant from Lemma 6.2. Furthermore,*

$$\|\Delta s(T)\| \leq \sqrt{1+\mu^2} \left|\frac{s_1(T)}{s_2(0)}\right| \|\Delta s(0)\|. \tag{22}$$

Given an exponent $0 < \alpha < 1$ and a parameter $0 < \mu < 1$ as in Lemma 6.3, define

$$\gamma = \frac{1}{2\alpha} + 2^{\alpha-1}(1+\mu) + \frac{1-\mu}{6} \quad \text{and} \quad \gamma' = \frac{1}{2\alpha} + \frac{1-\mu}{2^{\alpha+2}}. \tag{23}$$

Note $\gamma > \gamma' > 3$ for $0 < \alpha < 1/6$ and $0 < \mu < 1/2$.

Remark 6.4 The exponents γ and γ' will be used in defining $\gamma_1 = \gamma' - 2$ and $\gamma_2 = \gamma - 2$ in Theorem 3.1. The specific definitions of γ and γ' are needed to prove the technical estimates in Lemmas 6.5 and 6.6. The proofs of these lemmas are in [4], and are consequences of the estimates in Lemmas 6.2 and 6.3.

Lemma 6.5 (([4], Lemma 6.4)) *Under the assumptions of Lemma 6.3, there is a* $C_1 > 0$ *for which for any* $0 \leq t \leq T_1$,

$$|\Delta s_2(t)| \leq C_1 |\Delta s_2(0)| t^{-\gamma'}.$$

Lemma 6.6 (([4], Lemma 6.5)) *Under the assumptions of Lemma 6.3, one has*

$$\Delta s_2(t) \geq \left|\frac{\Delta s_2(0)}{s_2(0)}\right| |s_2(t)| \left(1 + C_0 |s_2(0)|^{2\alpha} t\right)^{-\beta_1}, \qquad 0 \leq t \leq T_1;$$

$$\Delta s_2(t) \geq \left|\frac{\Delta s_2(T_1)}{s_1(T_1)}\right| |s_1(t)| \left(1 + C_0 |s_1(T_1)|^{2\alpha}(t - T_1)\right)^{-\beta_2}, \qquad T_1 \leq t \leq T,$$

where

$$\beta_1 = (1+\mu)2^{\alpha-1} + \frac{1-\mu}{6} \quad \textit{and} \quad \beta_2 = \beta_1 + \frac{2\alpha}{\alpha}.$$

Lemma 6.7 *Under the assumptions of Lemma 6.3, there exists a* $C_2 > 0$ *for which for any* $0 \le t \le T_1$,

$$|\Delta s_2(t)| \ge C_2 |\Delta s_2(0)|, \qquad 0 < t < 1,$$
$$|\Delta s_2(t)| \ge C_2 |\Delta s_2(0)| t^{-\gamma}, \qquad t \ge 1.$$

Proof By inequality (a) in Lemma 6.2 and the first inequality in Lemma 6.6, for $0 < t < T_1$, we have:

$$\begin{aligned}\Delta s_2(t) &\ge \left|\frac{\Delta s_2(0)}{s_2(0)}\right| |s_2(t)| \left(1 + C_0 |s_2(0)|^{2\alpha} t\right)^{-\beta_1} \\ &\ge |\Delta s_2(0)| \left(1 + 2^\alpha C_0 |s_2(0)|^{2\alpha} t\right)^{-\beta_1 - 1/2\alpha}.\end{aligned}$$

For $0 < t < 1$, since $|s_2(0)| \le r_1$, we're done by setting

$$C_2 = \left(1 + 2^{(p-2)/p} C_0 r_1^{(2p-4)/p}\right)^{-\beta_1 - 1/2\alpha}.$$

For $t \ge 1$, since $1 + At \le (1 + A)t$ for $A > 0$, we have

$$|\Delta s_2(t)| \ge |\Delta s_2(0)| \left(1 + 2^\alpha C_0 |s_2(0)|^{2\alpha}\right)^{-\beta_1 - 1/2\alpha} t^{-\beta_1 - 1/2\alpha}.$$

Noting $\gamma = \beta_1 + \frac{1}{2\alpha}$, the same C_2 as in the $t < 1$ case satisfies the second estimate in the lemma. □

Lemma 6.8 (([4], Lemma 6.7)) *Under the assumptions of Lemma 6.3, there exist* $C_3, C_4 > 0$ *such that*

$$C_3 \Delta s_2(T_1) \ge \Delta s_2(T) \ge C_4 \Delta s_2(T_1).$$

7 A lower bound on the tail of the return time

Proving Theorem 3.1 requires polynomial upper and lower bounds on the tail of the return time, $\mu_1(\{x \in \Lambda : \tau(x) > n\})$ (where μ_1 is the g-invariant Riemannian measure from Proposition 4.4). We prove these bounds in this section and the next one.

To begin, we cite the following result, bounding the time a typical orbit stays near a singularity.

Lemma 7.1 ([7], Lemma 5.2) *There exists a* $T_0 \in \mathbb{Z}$*, depending on* r_0 *and* λ*, so that for any* $x \in \mathcal{U}_0 = \bigcup_{k=1}^m \phi_k^{-1}(D_{\rho_0})$*, we have*

$$\max\left\{N > 0 : g^n(x) \in \bigcup_{k=1}^m \phi_k^{-1}\left(D_{\rho_0} \setminus D_{\rho_1}\right) \text{ for all } n = 0, \dots N\right\} \le T_0.$$

Now, consider the Young structure on (M, g) constructed in Section 5 with stable sets Λ_s^i. Note the sets Λ_i^s consist of full-length stable curves through a Markov rectangle R. Fix one such curve σ. Denote $D_{\rho_j}^k = \phi_k^{-1}\left(D_{\rho_j}\right) \subset M$ for $j = 0, 1$.

Lemma 7.2 *Suppose a stable curve $\sigma \subset \Lambda_i^s$ enters a singular neighborhood $D_{\rho_1}^k$ at time $n > 1$, so that $g^n(\sigma) \cap D_{\rho_1}^k \neq \emptyset$, and that σ exits $D_{\rho_1}^k$ at time $m > n$. Then,*

$$C_5(m-n)^{-\gamma} \leq \frac{L(g^m(\sigma))}{L(g^n(\sigma))} \leq C_6(m-n)^{-\gamma'}, \tag{24}$$

where $C_5 > 0$, $C_6 > 0$ are constants independent of m, n, and the choice of stable curve σ; γ, γ' are as in (23); and L denotes the length of the curve.

Proof Let x, y be the endpoints of the curve γ in R. Set $x_k = g^k(x)$ and $y_k = g^k(y)$. Observe that there is a $K_0 > 0$ such that for all $k \geq 1$,

$$K_0^{-1} d(x_k, y_k) \leq L(g^k(\sigma)) \leq K_0 d(x_k, y_k). \tag{25}$$

where d is the Riemannian distance in M.

Let σ be as in the statement of the lemma. By assumptions on the pseudo-Anosov homeomorphism f, σ remains in a stable sector for the duration of time it remains in $D_{\rho_1}^k$ prior to exiting. In (s_1, s_2)-coordinates in the stable sector, it is enough to consider the map $g : M \to M$ near x_k to be the time-1 map $G_p : \mathbb{R}^2 \to \mathbb{R}^2$ of the vector field (5). (Recall $s_1 + is_2 = \left(\varphi^{-1} \circ \Phi_{kj} \circ \varphi_k\right)(x)$ for $x \in \mathcal{U}_0$; see (6), (9), and (10).)

Let $s, \tilde{s} : [0, m-n] \to \mathbb{R}^2$ be solutions to (5) with initial conditions $s(0) = \left(\varphi^{-1} \circ \Phi_{kj} \circ \phi_k\right)(x_n)$ and $\tilde{s}(0) = \left(\varphi^{-1} \circ \Phi_{kj} \circ \phi_k\right)(y_n)$, and note $s(0), \tilde{s}(0) \in D_{r_1}$ while $s(m-n), \tilde{s}(m-n)$ lies in $D_{r_0} \setminus D_{r_1}$. Also define $\Delta s_i(t) = \tilde{s}_i(t) - s_i(t)$ and let $\Delta s(t) = (\Delta s_1, \Delta s_2) \in \mathbb{R}^2$ be the difference vector from $\tilde{s}(t)$ to $s(t)$. Note that there is a K_1 independent of σ for which

$$K_1^{-1} \|\Delta s(j-n)\| \leq d(x_j, y_j) \leq K_1 \|\Delta s(j-n)\| \tag{26}$$

for all $n \leq j \leq m$.

We will apply Lemma 6.3. To check that the conditions are satisfied, first observe that Assumption 1 is satisfied since y is in the image of the stable cone of x under $\exp_x : T_x M \to M$. Assumption 2 is satisfied if $d(x_k, y_k)$, for $k = n, m$, is sufficiently small. This can be done, using Proposition 5.2 and (26), by taking $r_0 > 0$ in the construction of g so that $Q > 0$ is sufficiently large. So Lemma 6.3 applies.

Assume $\|\Delta s(0)\|$ is made sufficiently small so that the curve $G_p^j\left(\varphi^{-1} \circ \Phi_{kj} \circ \phi_k(\sigma)\right) \in \mathbb{R}^2$ lies in $D_{r_0/2} \cap \{(s_1, s_2) : s_1 > s_2\}$ for $n < j < \frac{n+m}{2}$ and lies in $D_{r_0/2} \cap \{(s_1, s_2) : s_1 < s_2\}$ for $\frac{n+m}{2} < j < m$. Applying Lemmas 6.3, 6.7, and 6.8 (with $T = m-n$ and $T_1 = (m-n)/2$), as well as (25) and (26), we obtain:

$$\begin{aligned}
L(g^m(\sigma)) &\geq K_0^{-1} d(x_m, y_m) \\
&\geq K_0^{-1} K_1^{-1} \|\Delta s(m-n)\| \\
&\geq K_0^{-1} K_1^{-1} |\Delta s_2(m-n)| \\
&\geq K_0^{-1} K_1^{-1} C_4 \left|\Delta s_2\left(\frac{m-n}{2}\right)\right| \\
&\geq K_0^{-1} K_1^{-1} C_2 C_4 |\Delta s_2(0)| \left(\frac{m-n}{2}\right)^{-\gamma} \\
&\geq K_0^{-1} K_1^{-1} C_2 C_4 2^{\gamma} (m-n)^{-\gamma} \frac{1}{\sqrt{1+\mu^2}} \|\Delta s(0)\| \\
&\geq K_0^{-2} K_1^{-2} C_2 C_4 2^{\gamma} (m-n)^{-\gamma} \frac{1}{\sqrt{1+\mu^2}} L(g^n(\sigma)).
\end{aligned}$$

The lower bound of (24) now follows with $C_5 = K_0^{-2}K_1^{-2}C_2C_4 2^\gamma/\sqrt{1+\mu^2}$.

To prove the upper bound, we use Lemmas 6.3, 6.5, and 6.8, as well as (25) and (26), to show:

$$\begin{aligned} L(g^m(\sigma)) &\le K_0 d(x_m, y_m) \\ &\le K_0K_1\|\Delta s(m-n)\| \\ &\le K_0K_1\sqrt{1+\mu}|\Delta s_2(m-n)| \\ &\le K_0K_1C_3\sqrt{1+\mu}\left|\Delta s_2\left(\frac{m-n}{2}\right)\right| \\ &\le K_0K_1C_1C_3 2^{\gamma'}\sqrt{1+\mu}|\Delta s_2(0)|(m-n)^{-\gamma'} \\ &\le K_0K_1C_1C_3 2^{\gamma'}\sqrt{1+\mu}(m-n)^{-\gamma'}\|\Delta s(0)\| \\ &\le K_0^2K_1^2C_1C_3 2^{\gamma'}\sqrt{1+\mu}(m-n)^{-\gamma'}L(g^n(\sigma)). \end{aligned}$$

The upper bound of (24) now follows with $C_6 = K_0^2K_1^2C_1C_3 2^{\gamma'}\sqrt{1+\mu}$. □

Let $\widetilde{\mathcal{P}}$ and $\mathcal{P}$ be Markov partitions for the pseudo-Anosov homemorphism f and the smooth realization g respectively, and let $\tilde{R} \in \tilde{\mathcal{P}}$ and $R \in \mathcal{P}$ be the partition elements discussed in Section 5.2. Fix the number Q as in Proposition 5.3. Assume the partition $\mathcal{P}$ and the numbers $0 < r_1 < r_0$ are chosen so that Proposition 5.2 holds. Finally, denote:

$$\mathcal{N} = \tau(\Lambda) = \{n \in \mathbb{N} : \text{there exists } x \in R \text{ such that } n = \tau(x)\}.$$

Lemma 7.3 *We may choose $\rho_0 > 0$ in the construction of g so that there is an integer $Q_0 > 0$ satisfying the following property: For each singularity x_k, for any $N > 0$, one can find $n \in \mathcal{N}$ with $n > N$, an s-subset Λ_l^s with $\tau(\Lambda_l^s) = n$ and numbers $0 < m_1 < m_2$ satisfying $m_1 < Q_0$, $n - m_2 < Q_0$ such that $g^l(\Lambda_l^s) \cap \mathcal{U}_0 = \emptyset$ for all $0 \le l < m_1$ and $m_2 < l \le n$, and $g^l(\Lambda_l^s) \cap \mathcal{U}_0 \ne \emptyset$ for all $m_1 \le l \le m_2$.*

Proof We will show that for each $k = 1, \dots, \ell$ (where ℓ is the number of singularities of f), there is an integer $Q_k > 0$ satisfying this proposition with $\mathcal{U}_0$ replaced with the neighborhood $D^k_{\rho_0}$ around the singularity x_k. Taking $Q_0 = \max\{Q_1, \dots, Q_m\}$ will yield the result.

Fix $k \in \{1, \dots, m\}$. To prove the existence of Q_k, it suffices to show there exists an integer $Q_k > 0$ such that for any $N > 0$, there is an admissible word of length $n > N$ of the form

$$R\bar{W}_1\bar{R}_k\bar{W}_2R, \tag{27}$$

where the words $\bar{W}_1$ and $\bar{W}_2$ are of length $|\bar{W}_q| < Q_k$ for $q = 1, 2$, and do not contain any of the symbols R or $R_{j,l}$ (the latter being elements of the Markov partition with a singularity x_j as a vertex; see Proposition 5.2), and the word $\bar{R}_k$ consists of one of the symbols $R_{k,1}, \dots, R_{k,2p(k)}$ repeated $|\bar{R}_k| = n - 2 - |\bar{W}_1| - |\bar{W}_2|$ times (since the stable and unstable sectors satisfy $g(S^{s/u}_{kj}) \subset S^{s/u}_{kj}$; see section 4.1). Observe that since this word of length n begins and ends with the symbol R, we have that $n \in \mathcal{N}$.

Because the smooth realization g is topologically conjugate to the linear pseudo-Anosov homeomorphism f, it suffices to prove that there is an admissible word of the form (27) for f and the Markov partition $\tilde{\mathcal{P}}$. To this end, consider the stable and unstable prongs through x_k. Since the singularities are fixed points by assumption, the prongs are invariant under f.

As before, let $P^s_{k,j} \subset D^k_{\rho_0}$ and $P^u_{k,j} \subset D^k_{\rho_0}$ $(1 \leq j \leq p(k))$ be the components of the stable and unstable prongs having x_k as an endpoint and contained in $D^k_{\rho_0}$. By topological transitivity of f ([29], Corollary 9.19), we know $f^q(\tilde{R}) \cap D^k_{\rho_0} \neq \emptyset$ for some integer $q \geq 1$ (recall $\tilde{R}$ is the Markov element of f, corresponding to the Markov element R of g under topological conjugacy). For each $j = 1, \ldots, p(k)$, there are minimal positive integers n^s_j and n^u_j for which $f^{-n^s_j}(P^s_{k,j}) \cap \tilde{R} \neq \emptyset$ and $f^{n^u_j}(P^u_{k,j}) \cap \tilde{R} \neq \emptyset$. For definiteness, without loss of generality, assume $n^s_1 = \max\left\{n^s_j : 1 \leq j \leq p(k)\right\}$, and let γ^s and γ^u accordingly be full-length stable and unstable curves in $\tilde{R}$ for which $P^s_{k,1} \supset f^{n^s_1}(\gamma^s)$ and $P^u_{k,1} \supset f^{-n^u_1}(\gamma^u)$. In particular, γ^s and γ^u are constructed so that they lie on stable and unstable manifolds that extend from stable and unstable prongs of x_k. By reducing ρ_0 if necessary (which we would need to do only finitely many times), we may assume $f^i(\gamma^s)$ and $f^{-i}(\gamma^u)$ enter $\mathcal{U}_0$ for the first and only time when $f^{n^s_1}(\gamma^s) \subset P^s_{k,1}$ and $f^{-n^u_1}(\gamma^u) \subset P^u_{k,1}$. It follows that $f^l(\gamma^s) \cap D^k_{\rho_0} = \emptyset$ for $0 \leq l < n^s_1$ and $f^{-l}(\gamma^u) \cap D^k_{\rho_0} = \emptyset$ for $0 \leq l < n^u_1$.

Since the manifolds extending the prongs are invariant under f, observe that $f^i(\gamma^s)$ and $f^{-i}(\gamma^u)$ never return to $\tilde{R}$ as $i \to \infty$. Thus, for any $n \in \mathcal{N}$ with $n > N$, there is a u-subset $\tilde{\Lambda}^u_{j_1}$ which completely enters $D^k_{\rho_0}$ at the same time as γ^u (iterated under f^{-1}), and an s-subset $\tilde{\Lambda}^s_{j_2}$ which completely enters $D^k_{\rho_0}$ at the same time as γ^s (iterated under f). Recall that $\gamma^s \subset \tilde{R}$ is an extension of the stable prong at the singularity x_k. Taking a point $x \in \tilde{\Lambda}^s_{j_2}$ sufficiently close to γ^s, we note that eventually $f^i(x) \in f^{-n^u_1}(\Lambda^u_{j_1})$, and so $f^{i+n^u_1}(x) \in \tilde{R}$. So the symbolic representation of x satisfies (27), with $Q_k = \max\{n^s_1, n^u_1\}$. This completes the proof of the lemma. □

Lemma 7.4 *There exists a constant $C_7 > 0$ such that*

$$\mu_1\left(\{x \in \Lambda : \tau(x) > n\}\right) > C_7 n^{-(\gamma-1)},$$

where μ_1 is the measure of Proposition 4.4 and γ is defined in (23).

Proof We begin by observing

$$\begin{aligned}
\mu_1\left(\{x \in \Lambda : \tau(x) > n\}\right) &= \sum_{N=n+1}^{\infty} \mu_1(\{x \in \Lambda : \tau(x) = N\}) \\
&= \sum_{N=n+1}^{\infty} \sum_{\Lambda^s_k : \tau(\Lambda^s_k) = N} \mu_1(\Lambda^s_k) \\
&> \sum_{N=n+1}^{\infty} \mu_1(\Lambda^s_l(N)),
\end{aligned}$$

where $\Lambda^s_l(N) =: \Lambda^s_l$ is the s-set defined in Lemma 7.3. We will show that there is a $K > 0$ for which

$$\mu_1(\Lambda^s_l(N)) \geq K N^{-\gamma} \tag{28}$$

for each $N \geq n+1$, where $\gamma > 0$ is given in (23). Once this is shown, we have

$$\mu_1(\{x \in \Lambda : \tau(x) > n\}) > \sum_{N=n+1}^{\infty} K N^{-\gamma} > C_7 n^{-(\gamma-1)}$$

for some constant $C_7 > 0$.

Given $x \in \Lambda_l^s$, let $\gamma_l^s(x) = \gamma^s(x) \cap \Lambda_l^s$ (where $\gamma^s(x)$ is the full-length stable leaf through x in the Markov rectangle R). Since the g-invariant measure μ_1 is determined locally by the product structure of the stable and unstable manifolds (by Lemma 4.2 and the definition of Ω_p), there is a constant $K_1 > 0$ independent of $x \in \Lambda_l^s$ such that

$$\mu_1(\Lambda_l^s) = \mu_1(g^N(\Lambda_l^s)) = K_1 L(g^N(\gamma_l^s(x))) \tag{29}$$

where L denotes the length of the curve.

Let $x_j = g^j(x)$ for $j = 0, \dots, N$. By Lemma 7.3, there are $k_1, k_2 \geq 1$ such that $g^j(x) \notin \mathcal{U}_0$ if $0 \leq j < k_1$ or if $k_2 < j \leq N$, and $g^j(x) \in \mathcal{U}_0$ if $k_1 \leq j \leq k_2$. Note that for $0 \leq j < k_1$ and $k_2 < j \leq N$, the curve $g^j(\gamma_l^s(x))$ lies in the stable cone for the pseudo-Anosov homeomorphism f at x_j, and indeed, is an admissible manifold for f (i.e., for $y \in g^j(\gamma_l^s(x))$, the tangent line $T_y g^j(\gamma_l^s(x))$ lies in the stable cone at y). Thus, the length of the curve $\gamma^s(x)$ contracts exponentially outside of the region $\mathcal{U}_0$ with contracting constant λ^{-1} (where we recall $\lambda > 1$ is the expansion constant for the pseudo-Anosov homeomorphism f). By the proof of Lemma 7.3, we have that $\gamma_l^s(x)$ enters and exits $\mathcal{U}_0$ at the same time as Λ_l^s, so $k_1 < Q_0$ and $N - k_2 < Q_0$. Therefore,

$$L(\gamma_l^s(x)) = \lambda^{k_1} L(g^{k_1}(\gamma_l^s(x))) \leq \lambda^{Q_0} L(g^{k_1}(\gamma_l^s(x))) \tag{30}$$

and

$$L(g^N(\gamma_l^s(x))) = \lambda^{-(N-k_2)} L(g^{k_2}(\gamma_l^s(x))) \geq \lambda^{-Q_0} L(g^{k_2}(\gamma_l^s(x))). \tag{31}$$

Let $\mathcal{U}_1 = \bigcup_{k=1}^m D_{\rho_1}^k$, where we recall m is the number of singularities and $D_{\rho_1}^k = \varphi_k^{-1}(D_{\rho_1})$ is the neighborhood of the singularity x_k given as the preimage of $D_{\rho_1} \subset \mathbb{C}$. By Lemma 7.1, the time the trajectory spends in $\mathcal{U}_0 \setminus \mathcal{U}_1$ is uniformly bounded. So by Lemma 7.2, there is a constant $\hat{C}_6 > 0$ such that

$$L(g^{k_2}(\gamma_l^s(x))) > \hat{C}_6 (k_2 - k_1)^{-\gamma} L(g^{k_1}(\gamma_l^s(x))). \tag{32}$$

Since $k_2 - k_1 < N$, by (29) - (32),

$$\begin{aligned}
\mu_1(\Lambda_l^s) &\geq K_1 L(g^N(\gamma_l^s(x))) \geq K_1 \lambda^{-Q_0} L(g^{k_2}(\gamma_l^s(x))) \\
&> K_1 \hat{C}_6 \lambda^{-Q_0} (k_2 - k_1)^{-\gamma} L(g^{k_1}(\gamma_l^s(x))) \\
&> K_1 \hat{C}_6 \lambda^{-2Q_0} L(\gamma_l^s(x)) N^{-\gamma}.
\end{aligned}$$

Note that since $\gamma_l^s(x)$ is a full-length stable curve in R, the length of $\gamma_l^s(x)$ is independent of N. So the value $K = K_1 C_6 \lambda^{-2Q_0} L(\gamma_l^s(x))$ is independent of N. This proves (28). ☐

8 An upper bound on the tail of the return time

We now prove that the tail of the return time of the Young structure of g has a polynomial upper bound. Recall that R is the element of the Markov partition of g containing the base of the Young tower, $\mathcal{U}_0$ is the ρ_0-neighborhood of the singularities, and $R \cap \mathcal{U}_0 = \emptyset$. Given an s-set $\Lambda_i^s \subset R$ with $\tau(\Lambda_i^s) = n$, choose integers $q = q(\Lambda_i^s)$ and $r = r(\Lambda_i^s)$, and two finite collections of numbers $\{k_j \geq 0\}_{j=1,\dots,q}$ and $\{l_j \geq 0\}_{j=0,\dots,q}$ such that

1. $k_1 + k_2 + \cdots + k_q = k$ and $l_0 + l_2 + \ldots + l_q = n - k$;
2. the trajectory of the set Λ_i^s under g^j, $0 \leq j \leq n$, consecutively spends l_q time outside $\mathcal{U}_0$ and k_q times inside $\mathcal{U}_0$.

Consider now the set of s-sets

$$\mathcal{S}_{k,n,q} = \{\Lambda_i^s \subset R : \tau(\Lambda_i^s) = n, k = k(\Lambda^s), q = q(\Lambda_s^i)\}.$$

Thus $\mathcal{S}_{k,n,q}$ is the set of s-sets with return time $\tau(\Lambda_i^s) = n$ and that spend a total of k time outside of $\mathcal{U}_0$ before returning to Λ_i^s, and enter $\mathcal{U}_0$ in total q times.

Lemma 8.1 *There are $0 < h < h_{\text{top}}(g)$, $\varepsilon_0 > 0$, and $C_8 > 0$ such that $\varepsilon_0 < h_{\text{top}}(g) - h$ and*

$$\#\mathcal{S}_{k,n,q} \leq C_8 \frac{1}{q^2} e^{(h+\varepsilon_0)(n-k)}. \tag{33}$$

Proof Recall that $\tilde{\mathcal{P}}$ is the Markov partition for the pseudo-Anosov homeomorphism $f : M \to M$, and that $H : M \to M$ is the conjugacy map between the pseudo-Anosov homeomorphism f and its smooth model g, so $g \circ H = H \circ g$. Further, recall that for each singularity x_l, $l = 1, \ldots, m$, the Markov rectangle $\tilde{R}_{j,p(l)} \in \tilde{\mathcal{P}}$, for $1 \leq j \leq 2p(l)$, is one of the $2p(k)$ rectangles with the singularity x_k as a vertex. Let $R_{j,p(l)} = H(\tilde{R}_{j,p(l)})$. Define the set $V = \bigcup_{l=1}^{m} \bigcup_{j=1}^{2p(l)} R_{j,l}$, and the number

$$s := \sum_{l=1}^{m} 2p(l)$$

to be the number of Markov rectangles making up V, i.e., the number of Markov rectangles with a vertex containing a singularity.

Observe that for a particular $\Lambda_i^s \in \mathcal{S}_{k,n,q}$, the symbolic representation of every $x \in \Lambda_i^s$ has the same first $n = \tau(\Lambda_i^s)$ symbols, which begin and end with R. By (19), it follows that the cardinality of $\mathcal{S}_{k,n,q}$ is less than or equal to the set of all words of length n that begin and end with R, and which contain k instances of the symbols $R_{j,p(l)}$, $1 \leq l \leq m$, $1 \leq j \leq 2p(l)$, and for which the remaining $n - k$ symbols do not have singularities in their closures. We will show that the number of such words is bounded by (33).

Given k and q, the number of ways k can be partitioned into q summands respecting order is $\binom{k-1}{q-1}$, and so the number of ways the orbit of $x \in \Lambda_i^s$ can enter the set V p times with total time in V not exceeding k is $\leq \binom{k-1}{q-1}$. Likewise, the number of ways $n - k$ can be partitioned into $q + 1$ summands respecting order is $\binom{n-k-1}{q}$, and so the number of ways the orbit of $x \in \Lambda_i^s$ can enter V^c (counting the "zeroth" entry when it starts in R) without exceeding $n - k$ is $\leq \binom{n-k-1}{q}$. So there are $\binom{k-1}{q-1}\binom{n-k-1}{q}$ pairs of ordered sets of integers $(k_1, \ldots, k_q)$, $(l_0, \ldots, l_q)$ for which $k_1 + \cdots + k_q = k$ and $l_0 + \cdots + l_q = n - k$.

Consider one such pair of ordered sets $(k_1, \ldots, k_q)$, $(l_0, \ldots, l_q)$. By assumption, the map f (and thus the map g) preserve the stable sectors S_{jl}^s around each singularity x_l. Assume the Markov partition is sufficiently small so that each rectangle is contained in one of the coordinate charts U_j defining the homeomorphism f. It follows that when the orbit of $x \in \Lambda_i^s$ enters V the r^{th} time, its symbolic representation contains k_r copies of a single symbol $R_{j,p(l)}$. Therefore, the first n-letter word in the symbolic representation of an $x \in \Lambda_i^s$ with times $(k_1, \ldots, k_q)$ spent in V and times $(l_0, \ldots, l_q)$ spent outside of V, is of the form

$$\underline{R}_{l_0} [R_{j(1),l(1)}]^{k_1} \underline{R}_{l_1} [R_{j(2),l(2)}]^{k_2} \cdots [R_{j(q),l(q)}]^{k_q} \underline{R}_{l_q}$$

where each $\underline{R}_{l_r}$ is a word in $\mathcal{P}$ of length l_r not including letters in V, and $[R_{j(r),l(r)}]^{k_r}$ is a word made of k copies of $R_{j(r),l(r)}$. Observe that for each $(k_1,\ldots,k_q)$, there are s^q possible configurations of $[R_{j(1),l(1)}]^{k_1},\ldots,[R_{j(q),l(q)}]^{k_q}$.

Now consider a word of length l_r. Given a topologically mixing topological Markov shift $\sigma:\mathcal{A}^{\mathbb{Z}}\to\mathcal{A}^{\mathbb{Z}}$ over an alphabet $\mathcal{A}$, and a set $\mathcal{B}\subset\mathcal{A}$ of forbidden letters, there is a $C>0$ and an $h\in(0,h_{\text{top}}(\sigma))$ for which the number of words of length n not including any symbols from $\mathcal{B}$ is $\leq Ce^{nh}$. Since the Markov shift associated to $g:M\to M$ with symbols $\mathcal{P}$ is topologically mixing (as all pseudo-Anosov homeomorphisms on surfaces are topologically transitive), it follows that the number of words of length l_r not including the symbols in $\{R_{j,l}\}$ is $\leq C_8e^{hl_r}$, where $C_8>0$ is independent of l_r and $h<h_{\text{top}}(g)=h_{\text{top}}(f)$. So to summarize, for $k,n,q\geq 1$, there are $\binom{k-1}{q-1}\binom{n-k-1}{q}$ possible pairs of ordered sets $(k_1,\ldots,k_q)$, $(l_0,\ldots,l_q)$ with $k_1+\cdots+k_q=k$, $l_0+\ldots+l_q=n-k$; for each such ordered set $(k_1,\ldots,k_q)$, there are s^q possible configurations of the $R_{j(r),l(r)}$, $1\leq r\leq q$; and for each l_r, there are $\leq C_8e^{hl_r}$ possible words of length l_r. Since $l_0+\cdots+l_q=n-k$, it follows that

$$\begin{aligned}\#\mathcal{S}_{k,n,q}&\leq C_8s^q\binom{k-1}{q-1}\binom{n-k-1}{q}e^{h(n-k)}\\&=\frac{C_8}{q^2}q^2s^q\binom{k-1}{q-1}\binom{n-k-1}{q}e^{h(n-k)}.\end{aligned}\tag{34}$$

Our goal is to estimate the quantity $q^2s^q\binom{k-1}{q-1}\binom{n-k-1}{q}$.

We begin by bounding $\binom{k-1}{q-1}$. By Proposition 5.2, it takes Λ_i^s at least Q iterates before it reenters $\mathcal{U}_{\rho_0}$ after exiting (or after starting from the rectangle R). This means $n=k+l_0+\ldots+l_q>k+(q+1)Q$, i.e., $q+1<\frac{n-k}{Q}$. Now, for a fixed k, $\binom{k-1}{q-1}$ is maximized when $q-1=\left\lfloor\frac{k-1}{2}\right\rfloor$. It follows that $\left\lfloor\frac{k-1}{2}\right\rfloor<\frac{n-k}{Q}$. Using the asymptotic formula $\binom{a}{b}<\left(\frac{ae}{b}\right)^b$, we obtain:

$$\begin{aligned}\binom{k-1}{q-1}&\leq\binom{k-1}{\left\lfloor\frac{k-1}{2}\right\rfloor}\\&<\left(\frac{(k-1)e}{\left\lfloor\frac{k-1}{2}\right\rfloor}\right)^{\lfloor(k-1)/2\rfloor}\\&\leq(2e)^{\lfloor(k-1)/2\rfloor}\\&<(2e)^{\frac{n-k}{Q}}\\&<e^{\frac{n-k}{Q}\ln(2e)}.\end{aligned}\tag{35}$$

Next we estimate $\binom{n-k-1}{q}$. Note $q<\min\left\{\frac{k-1}{2},\frac{n-k}{Q}\right\}$, and so using the asymptotic formula from earlier, we observe:

$$\binom{n-k-1}{q}<\binom{n-k}{\left\lfloor\frac{n-k}{Q}\right\rfloor}<\left(\frac{(n-k)e}{\frac{n-k}{Q}}\right)^{\frac{n-k}{Q}}<e^{\frac{n-k}{Q}\ln\frac{(n-k)e}{(n-k)/Q}}=e^{\frac{n-k}{Q}\ln(Qe)}.\tag{36}$$

Finally, observe:

$$q^2s^q=e^{2\ln q+q\ln s}<e^{2q+q\ln s}<e^{\frac{n-k}{Q}(2+\ln s)}.\tag{37}$$

Given sufficiently small $\varepsilon_0>0$, one can choose Q large enough so that

$$\frac{1}{Q}\left(2+\ln s+\ln(2e)+\ln(Qe)\right)<\varepsilon_0.$$

Applying this to the estimates (35), (36), and (37), we obtain:

$$q^2 s^q \binom{k-1}{q-1}\binom{n-k-1}{q} e^{h(n-k)} \leq e^{(n-k)(h+\varepsilon_0)}$$

and therefore, from (34),

$$\#\mathcal{S}_{k,n,q} \leq \frac{C_8}{q^2} e^{(h+\varepsilon_0)(n-k)}.$$

□

Lemma 8.2 *There is an $\varepsilon_0 > 0$ such that for any $\Lambda_i^s \in \mathcal{S}_{k,n,q}$,*

$$\mu_1\left(\Lambda_i^s\right) \leq C_9 k^{-\gamma'} e^{(-\log\lambda+\varepsilon_0)(n-k)},$$

where $C_9 > 0$ is a constant and γ' is given in (23).

Proof Let $x \in \Lambda_i^s$, and let $\gamma_i^s(x) = \gamma_i^s \subset \Lambda_i^s$ be the connected component of the stable manifold of x intersected with Λ_i^s that contains x. By (29), we have $\mu_1(\Lambda_i^s) = K_1 L(g^n(\gamma_i^s(x)))$. Note further that the length of the backwards iterates of γ_i^s lying outside of $\mathcal{U}_0$ are stretched by the expansion factor λ of the pseudo-Anosov homeomorphism f. Additionally, the time spent in $\mathcal{U}_0 \setminus \mathcal{U}_1$ is uniformly bounded by Lemma 7.1, and therefore whenever the orbit of γ_i^s enters $\mathcal{U}_0$, we can use Lemmas 7.1 and 7.2 to give an upper bound for its length. So, letting $\{k_j \geq 0\}_{j=1,\dots,q}$ and $\{l_j \geq 0\}_{j=0,\dots,q}$ be such that the orbit of γ_i^s spends k_j times consecutively inside $\mathcal{U}_0$ and l_j times outside $\mathcal{U}_0$, we obtain:

$$\begin{aligned}
\mu_1(\Lambda_i^s) &= K_1 L(g^n(\gamma_i^s)) \\
&\leq K_1 \lambda^{-l_q} L(g^{n-l_q}(\gamma_i^s)) \\
&\leq K_1 C_6 k_q^{-\gamma'} \lambda^{-l_q} L(g^{n-l_q-k_q}) \\
&\;\;\vdots \\
&\leq K_1 C_6^q \lambda^{-(l_q+\cdots+l_0)} \left(k_q k_{q-1}\cdots k_1\right)^{-\gamma'} L(\gamma_i^s).
\end{aligned} \tag{38}$$

Since $\gamma_i^s(x)$ is a full-length stable curve in R, its length is independent of $n = \tau(\Lambda_i^s)$. So we may take $K_1 L(\gamma_i^s) \leq K_1'$ for some $K_1' > 0$. Furthermore, if ρ_0 is made sufficiently small, the time k_i that the orbit stays in $\mathcal{U}_0$ may be made to be ≥ 2. Therefore,

$$k_1 k_2 \cdots k_q \geq 2^{q-1} \max_{1\leq i\leq q} k_i \geq q \max_{1\leq i\leq q} k_i \geq \sum_{i=1}^{q} k_i = k. \tag{39}$$

Finally, $C_6^q = e^{q\ln C_6} < e^{\frac{n-k}{Q}\ln C_6} < e^{\varepsilon_0(n-k)}$ for sufficiently small $\varepsilon_0 > 0$ and sufficiently large $Q \geq 1$. Therefore, applying this estimate and (39) to (38), we obtain:

$$\mu_1(\Lambda_i^s) < K_1' e^{\varepsilon_0(n-k)} \lambda^{-(n-k)} k^{-\gamma'} < C_9 k^{-\gamma'} e^{(-\log\lambda+\varepsilon_0)(n-k)}.$$

□

Lemma 8.3 *There exists a constant $C_{10} > 0$ such that*

$$\mu_1(\{x \in \Lambda : \tau(x) > n\}) < C_{10} n^{-(\gamma'-1)},$$

where $\gamma' > 0$ is defined in (23).

Proof Observe that:

$$\mu_1(\{x \in \Lambda : \tau(x) = n\}) \leq \sum_{k=1}^{n} \sum_{q=1}^{k} \left(\max_{\Lambda_i^s \in \mathcal{S}_{k,n,q}} \mu_1\left(\Lambda_i^s\right) \right) \#\mathcal{S}_{k,n,q}.$$

It follows from Lemmas 8.1 and 8.2 that:

$$\mu_1(\{x \in \Lambda : \tau(x) = n\}) \leq \sum_{k=1}^{n} \sum_{q=1}^{k} C_8 C_9 \frac{1}{q^2} k^{-\gamma'} e^{(-\log\lambda + \varepsilon_0)(n-k)} e^{(h+\varepsilon_0)(n-k)}$$

$$< C_8 C_9 \frac{\pi^2}{6} e^{-\delta n} \sum_{k=1}^{n} k^{-\gamma'} e^{\delta k}$$

where $\delta = \log\lambda - h - 2\varepsilon_0 > 0$ if $\varepsilon_0 > 0$ is sufficiently small.

To estimate $\sum_{k=1}^{n} k^{-\gamma'} e^{\delta k}$, let $u_k = k^{-\gamma'} e^{\delta k}$, and note that:

$$u_{k+1} - u_k = e^{\delta k} k^{-\gamma'} \left(e^{\delta} \left(\frac{k}{k+1} \right)^{\gamma'} - 1 \right) \sim e^{\delta k} k^{-\gamma'},$$

where $a_k \sim b_k$ means $\lim_{k\to\infty} \frac{a_k}{b_k}$ exists and is > 0 for positive sequences a_k and b_k. It follows that:

$$\sum_{k=1}^{n} u_k \sim \sum_{k=1}^{n} u_{k+1} - u_k = u_{n+1} - u_1 \sim e^{\delta n} n^{-\gamma'},$$

where the first asymptotic comparison comes from the Stolz-Cesàreo theorem, since $u_k > 0$ for all k and the series $\sum_{k=1}^{\infty} u_k$ diverges. Therefore, there is a $C_9' > 0$ for which

$$\mu_1(\{x \in \Lambda : \tau(x) = n\}) \leq C_8 C_9 \frac{\pi^2}{6} e^{-\delta n} \sum_{k=1}^{n} u_k \leq C_9' n^{-\gamma'}.$$

It follows that there is a $C_{10} > 0$ independent of n for which:

$$\mu_1(\{x \in \Lambda : \tau(x) > n\}) = \sum_{k>n} \mu_1(\{x \in \Lambda : \tau(x) = k\}) < C_{10} n^{-(\gamma'-1)}.$$

This concludes the proof of the Lemma and the upper bound on the tail of the return time. □

9 Proof of Theorem 3.1

We now prove the main result. Statements (1) and (2) of Theorem 3.1 are shown in Propositions 4.4 and 4.7. Statement (3), which gives the Bernoulli property of the diffeomorphism g with respect to the SRB measure μ_1, is a straightforward consequence of the Bernoulli property of the pseudo-Anosov homeomorphism f with respect to its SRB ν (the existence of which is given in Proposition 2.6). Indeed, note ν is absolutely continuous with respect to μ_1. The conjugating homeomorphism $h : M \to M$ for which $f = h \circ g \circ h^{-1}$ is C^1 away from the singularities of f; in particular, h transfers (un)stable manifolds of g to (un)stable manifolds of f. It follows that the measure $h_*\mu_1$ is an f-invariant SRB measure. Since f is topologically transitive, its SRB measure is unique by [32], so $h_*\mu_1 = \nu$. Since (M, f, ν) is Bernoulli by [29], and f and g are measure-theoretically isomorphic, g is also Bernoulli. This proves Statement (3). Statement (7) of Theorem 3.1 follows from Proposition 4.8, Statement (1), when $t = 0$.

In the remainder of the section, we will use the Young tower on $g : M \to M$ to show that g has polynomial upper and lower bounds on the decay of correlations (Statement (4)), that g satisfies the CLT (Statement (5)), and that g has polynomial large deviations (Statement (6)).

9.1 Decay of correlations

By Proposition 5.3, the pseudo-Anosov smooth model $g : M \to M$ is a Young diffeomorphism with base Λ, s-sets Λ_i^s, and inducing times $\tau = \{\tau_i\}$, $\tau_i = \tau(\Lambda_i^s)$. The associated *Young tower* is the space

$$\hat{Y} = \{(x, k) \in \Lambda \times \mathbb{N}_0 : 0 \leq k < \tau(x)\}$$

and the associated map $\hat{g} : \hat{Y} \to \hat{Y}$ is given by

$$g(x, k) = \begin{cases} (x, k+1) & \text{if } 0 \leq k < \tau(x) - 1, \\ (g(x), 0) & \text{if } k = \tau(x) - 1. \end{cases}$$

Define the subsets $\hat{M}_k \subset \hat{Y}$ by

$$\hat{M}_k = \{(x, \ell) \in \hat{Y} : 0 \leq \ell \leq \min\{k, \tau(x)\}\}.$$

Note $\hat{M}_k$ is the set of the first k levels of the Young tower. Finally define the projection $\pi : \hat{Y} \to M$ by $\pi(x, \ell) = g^\ell(x)$, and define

$$Y = \pi(\hat{Y}), \quad M_k = \pi(\hat{M}_k).$$

Note that the sets M_k are nested and exhaust Y.

Proving that $g : M \to M$ admits upper and lower bounds on polynomial decay of correlations requires the following result, which follows from Theorem 2.3 in [31] and Theorem 7.1 in [30] and its proof:

Proposition 9.1 *Let μ_1 be the SRB measure of $g : M \to M$. Assume that:*

- *the greatest common divisor of the inducing times $\tau_i = \tau(\Lambda_i^s)$ is 1;*
- *there is a constant $C > 0$ for which for all $\Lambda_i^s \subset \Lambda$, all $x, y \in \Lambda_i^s$, and all $0 \leq k \leq \tau_i$,*

$$d(f^k(x), f^k(y)) \leq C \max\{d(x, y), d(f^{\tau_i}(x), f^{\tau_i}(y))\};$$

- *there is a constant $\theta > 1$ such that $\mu_1(\tau > n) = O(n^{-\theta})$.*

Then the following statements hold:

1. $\mathrm{Cor}_n(h_1, h_2) = O(n^{1-\theta})$ *for any $h_1, h_2 \in C^\eta(M)$.*
2. *For any $h_1, h_2 \in C^\eta(M)$ supported in M_k for some $k > 0$, we have:*

$$\mathrm{Cor}_n(h_1, h_2) = \sum_{k=n+1}^{\infty} \mu_1(\tau(x) > k) \int h_1 \, d\mu_1 \int h_2 \, d\mu_1 + O(R_\theta(n)), \quad (40)$$

where:

$$R_\theta(n) = \begin{cases} n^{-\theta} & \text{if } \theta > 2, \\ n^{-2}\log n & \text{if } \theta = 2, \\ n^{-2(\theta-1)} & \text{if } 1 < \theta < 2. \end{cases}$$

Moreover, if $\int h_1\, d\mu_1 \int h_2\, d\mu_1 = 0$*, then* $\mathrm{Cor}_n(h_1,h_2) = O(n^{-\theta})$.

Remark 9.2 The consequences in Proposition 9.1 are the same as in Theorem 2.3 in [31]. There, the authors prove these results in higher generality for equilibrium states for a given potential; however, in addition to the above assumptions, the potential is assumed to satisfy certain conditions ((P1) - (P4) in [31] and [30]). In the proof of Theorem 7.1 in [30], it is shown that the geometric potential $\varphi(x) = -\log \left| dg|_{E^u(x)} \right|$ of a Young diffeomorphism g satisfies conditions (P1) - (P4) in [31]. Since the pseudo-Anosov smooth model $g : M \to M$ is a Young diffeomorphism, it remains only to verify the assumptions in Proposition 9.1 to apply the result to the pseudo-Anosov smooth model g.

Remark 9.3 Below, we assume that the parameter $\alpha > 0$ in the construction of g is taken to be $\alpha < 1/6$. However, to prove polynomial decay of correlations, it is only needed that $\alpha < 1/4$. Indeed, by (23), if $\alpha < 1/4$, then $\gamma' > 2$, in which case $\gamma_1 = \gamma' - 2 > 0$, and we have polynomial decay. We will require, however, that $\gamma_1 > 1$ in order to prove the central limit theorem below. For that, we assume $\alpha < 1/6$.

Proof of Theorem 3.1, (4) We begin by proving the upper bound (statement (a) of Theorem 3.1, (4)). By Proposition 9.1, the claim is immediate once we verify the three conditions of the proposition.

First, recall that g is topologically conjugate to the pseudo-Anosov homeomorphism f. Since f is Bernoulli [29], every power of f is ergodic. If $\tilde{\Lambda} = \bigcup_{i\geq 1} \tilde{\Lambda}_i^s$ is the base of the Young structure for f and the inducing times are $\tilde{\tau} : \tilde{\Lambda} \to \mathbb{N}_0$ (see Section 5.2), then $\gcd(\tilde{\tau}_i) = 1$ (where $\tilde{\tau}_i = \tilde{\tau}(\tilde{\Lambda}_i^s)$), and so $\gcd(\tau_i) = 1$.

The second assumption of Proposition 9.1 follows from the fact that $g : M \to M$ has a Young structure, and the images $g^k(\Lambda_i^s)$, $1 \leq k \leq \tau_i - 1$, have diameter less than the diameter of the Markov partition.

Finally, the third assumption holds because by Lemmas 7.4 and 8.3, we have

$$\frac{C_8}{n^{\gamma-1}} < \mu_1(\{x \in \Lambda : \tau(x) > n\}) < \frac{C_{10}}{n^{\gamma'-1}} \tag{41}$$

where γ, γ' are defined in (23). Note for $0 < \alpha < \frac{1}{6}$ and $0 < \mu < \frac{1}{2}$ we have that $\gamma > \gamma' > 3$, so the third assumption holds. Statement (a) of Proposition 9.1 gives the upper bound on the decay of correlations (statement (4)(a) of Theorem 3.1), using $\gamma_1 = \gamma' - 2 > 1$.

To prove the lower bound, by Statement (b) of Proposition 9.1 with $\theta = \gamma' - 1 > 2$, we have that for all $h_1, h_2 \in C^\eta(M)$ supported in M_k, for some $k > 0$:

$$\mathrm{Cor}_n(h_1,h_2) = \sum_{\ell=n+1}^{\infty} \mu_1(\{x : \tau(x) > \ell\}) \int_M h_1\, d\mu_1 \int_M h_2\, d\mu_1 + O(n^{-\gamma'+1}). \tag{42}$$

By the assumption on h_1, h_2 in Statement (4)(b) of Theorem 3.1, we may apply (41) and (42) above and obtain:

$$\mathrm{Cor}_n(h_1, h_2) > \sum_{\ell=n+1}^{\infty} K_1' \ell^{-(\gamma-1)} + K_2 n^{-(\gamma'-1)} > K_1 n^{-(\gamma-2)} + K_2 n^{-(\gamma'-1)} \tag{43}$$

for constants K_1', K_1, and K_2 (depending on h_1, h_2). By the definitions of γ and γ' in Equation (23), after choosing $0 < \mu < \frac{1}{2}$, we can show $\gamma - 2 < \gamma' - 1$ for all $0 < \alpha < \frac{1}{6}$. So there is a $C > 0$ for which

$$\mathrm{Cor}_n(h_1, h_2) > \frac{C}{n^{\gamma-2}}.$$

□

9.2 The Central Limit Theorem

Proof of Theorem 3.1, (5) By Statement (4)(a) of Theorem 3.1, if $h \in C^\eta$ satisfies $\int h\, d\mu_1 = 0$, then $\mathrm{Cor}_n(h, h) = O(n^{-(\gamma'-2)})$. Therefore the correlation function is summable for $\gamma' > 3$. It follows from Theorem 1.2 of [33] (see also Section 3 of [33]) that the system (M, g, μ_1) has the central limit theorem with respect to Hölder potentials. □

9.3 Large Deviations

Proof of Theorem 3.1, (6) Because the map $g : M \to M$ is modeled by a Young tower, the upper bound in (41) allows us to use Theorem 4.2 in [16] to show that for $0 < \alpha < \frac{1}{6}$ (so that $\gamma' > 3$), and for all sufficiently small $a > 0$ and all Hölder $h : M \to \mathbb{R}$, there is a constant $C = C_{h,a}$ depending continuously on h (in the C^η topology) such that for all $\varepsilon > 0$ and all sufficiently large $n \geq 0$,

$$\mu_1 \left(\left\{ \left| \frac{1}{n} \sum_{i=0}^{n-1} h(g^i(x)) - \int h\, d\mu_1 \right| > \varepsilon \right\} \right) < C_{h,a} \varepsilon^{-2(\gamma'-2-a)} n^{-(\gamma'-2-a)}.$$

This proves the first part of Statment (6) of Theorem 3.1. To obtain a lower bound on the large deviations, we will use Theorem 4.3 in [16]. The one condition of this theorem that needs to be checked is that $\mu_1(\overline{M}_k') < 1$ for some $k \geq 0$, where we have $M_k' = \pi(\hat{Y}_k)$ and

$$\hat{Y}_k = \{(x, \ell) \in \hat{Y} : \tau(x) \geq k\}$$

and $\pi : \hat{Y} \to M$ is the projection $\pi(x, k) = f^k(x)$ for $x \in \Lambda$, $0 \leq k \leq \tau(x) - 1$.

Given $k \geq 0$, choose a partition element Λ_i^s in the base Λ of the Young tower with $\tau(\Lambda_i^s) \leq k$. Identifying Λ_i^s with a subset of the 0-level of the tower $\hat{Y}$, we see $\Lambda_i^s \subset \hat{Y} \setminus \hat{Y}_k$, and we see that $\hat{\mu}_1(\Lambda_i^s) > 0$, where $\hat{\mu}_1$ is the lifted measure of μ_1 to the tower $\hat{Y}$. It follows that $\hat{\mu}_1(\hat{Y}_k) < 1$, and since the projection $\pi : \hat{Y} \to M$ is measure-preserving, $\mu_1(M_k) < 1$. So, by Theorem 4.3 in [16], for any $a > 0$, there is an open and dense subset of Hölder observables h for which for all $\varepsilon > 0$ sufficiently small, and infinitely many n, we obtain the lower bound

$$n^{-(\gamma'-2+a)} < \mu_1 \left(\left\{ \left| \frac{1}{n} \sum_{i=0}^{n-1} h(g^i(x)) - \int h\, d\mu_1 \right| > \varepsilon \right\} \right).$$

□

Supplementary information. Not applicable.

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Declarations

Not applicable.
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An appendix contains supplementary information that is not an essential part of the text itself but which may be helpful in providing a more comprehensive understanding of the research problem or it is information that is too cumbersome to be included in the body of the paper.

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