SRB measures of singular hyperbolic attractors

PSU Workshop on Dynamical Systems and Related Topics

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October 31 2020

Singular hyperbolic attractors

Setting:

- M Riemannian manifold, $K \subset M$ open and precompact, $N \subset K$ closed, $N^+ = N \cup \partial K$;
- $f : K \setminus N \to K$ diffeomorphism onto its image;
- $K^+ = \{x \in K : f^n(x) \notin N^+ \ \forall n \geq 0\}$;
- $D = \bigcap_{n \geq 0} f^n(K^+)$, $\Lambda = \overline{D}$ (Λ is the *attractor* for f).
- Λ is a *singular hyperbolic attractor* if there is a continuous splitting $z \mapsto E^s(z) \oplus E^u(z)$ over $K \setminus N$ into *stable* and *unstable* subspaces. In particular, there are $C > 0$ and $\lambda > 1$ so that for any $z \in D$, $n \geq 0$:

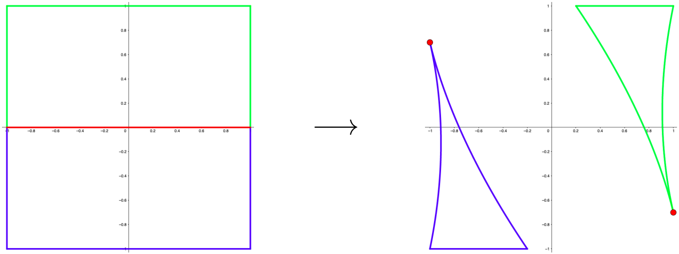
$$\begin{aligned} \|df_z^n v\| &\leq C\lambda^{-n} \|v\| \quad \forall v \in E^s(z); \\ \left\|df_z^{-n} v\right\| &\leq C\lambda^{-n} \|v\| \quad \forall v \in E^u(z). \end{aligned}$$

The geometric Lorenz attractor

- $I = (-1, 1)$, $K = I \times I$, $N = I \times \{0\}$, $f : K \setminus N \to K$ given by $f(x, y) = \big(\varphi(x, y), \psi(x, y)\big)$, where

$$\varphi(x, y) = (\operatorname{sgn}(y) B x |y|^{\nu} - B|y|^{\nu_0} + 1) \operatorname{sgn}(y)$$
$$\psi(x, y) = ((1 + A)|y|^{\nu_0} - A) \operatorname{sgn}(y)$$

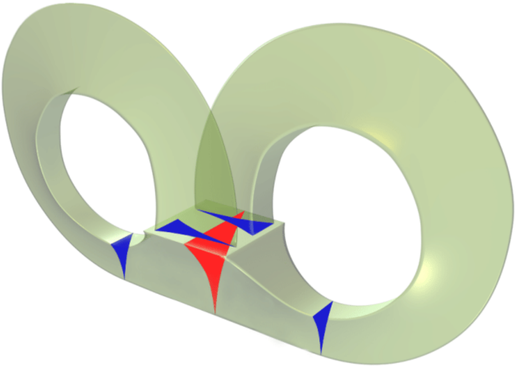
where $0 < A < 1$, $0 < B < \frac{1}{2}$, $\nu > 1$, and $1/(1 + A) < \nu_0 < 1$.



- The two dots form the set N^-, the "image" of the singular set N.

Geometric Lorenz attractor

The geometric Lorenz attractor is a Poincaré first-return map for a transverse cross-section of the flow of the classical Lorenz attractor.



Lorenz-type maps

More generally, a *Lorenz-type map* is a map $f : K \setminus N \to K$, $K = I \times I$, $N = I \times \{a_0, a_1, \ldots, a_{q+1}\}$, where

$$-1 = a_0 < a_1 < \cdots < a_q < a_{q+1} = 1$$

and, in addition to certain regularity conditions on df,

- $\lim_{y \uparrow a_i} f(x,y) = f_i^-$, $\lim_{y \downarrow a_i} f(x,y) = f_i^+$ ($f_i^\pm \in \overline{K \setminus N}$ constant points, independent of $x \in I$);
- $f|_{I \times (a_i, a_{i+1})} : I \times (a_i, a_{i+1}) \to K$ is a diffeomorphism onto its image;
- $\|\varphi_x\|, \left\|\psi_y^{-1}\right\| < 1$, where $f(x,y) = \big(\varphi(x,y), \psi(x,y)\big)$.

Theorem (Afraimovich, Bykov, Shilnikov '83)

If M is a compact Riemannian manifold w/ $\dim M \geq 3$, there exists a vector field X and a smooth submanifold S such that the first-return time map f induced on S by the flow given by X is a Lorenz-type map.

SRB measures

- Suppose $f : U \to M$ is a hyperbolic map on a Riemannian manifold M. An *SRB measure* is an invariant Borel probability measure μ for which:
    - f has positive Lyapunov exponents μ-a.e., and
    - μ admits absolutely continuous conditional measures on the unstable leaves (w.r.t. Riemannian leaf volume)
- SRB measures are hyperbolic *physical measures*:

$$m\left\{x : \frac{1}{n}\sum_{k=0}^{n-1}\left(\varphi \circ f^k\right)(x) \xrightarrow{n\to\infty} \int_U \varphi \, d\mu \quad \forall \varphi \in C^0\right\} > 0$$

where m is the Lebesgue/Riemannian volume.

- If $f : M \to M$ is a hyperbolic diffeomorphism (i.e., Anosov or pseudo-Anosov map), SRB measure is Riemannian volume.
- For dissipative dynamical systems (i.e., hyperbolic attractors), SRB measure is supported on attractor.

SRB measures for hyperbolic maps

Theorem (Rodriguez-Hertz, Rodriguez-Hertz, Tahzibi, Ures '10)

If $f : M \to M$ is a topologically transitive $C^{1+\alpha}$ diffeomorphism, then it admits at most one SRB measure.

Theorem (Pesin '92)

Suppose $f : K \setminus N \to K$ admits a singular hyperbolic attractor Λ. Then there are at most countably many ergodic SRB measures supported on Λ.

Setting of main result

Setting:

- M Riemannian manifold, $K \subset M$ open and precompact, $N \subset K$ closed;
- $f : K \setminus N \to K$ diffeomorphism onto its image;
- $N^- =$ image of continuous extensions of f to $N^+ \subset \overline{K}$; or more formally,

$$N^- = \{y \in K : \exists z \in N^+, z_n \in K \setminus N \text{ s.t. } z_n \to z,\ f(z_n) \to y\}$$

(for example, $N^- = \{f_i^\pm : 1 \le i \le q\}$ for Lorenz-type maps, where $f_i^+ = \lim_{y \downarrow a_i} f(x,y)$ and $f_i^- = \lim_{y \uparrow a_i} f(x,y)$);

- Λ a singular hyperbolic attractor, expansive constant $\lambda > 1$.

Main result

Assumptions:

1. N is a disjoint union of finitely many closed submanifolds $N_1, \ldots, N_m$ with boundary;
2. $f^j(N^-) \cap N = \varnothing$ for $0 \leq j < k$, $\lambda^k > 2$.

Theorem (V.)

If $f : K \setminus N \to K$ as above satisfies these assumptions, then the attractor Λ admits finitely many ergodic SRB measures.

Corollary

Let $f : K \setminus N \to K$ be a Lorenz-type map. Suppose there are $C_i > 0$ and sufficiently small $\gamma > 0$ for which

$$\rho(f^n(f_i^{\pm}), N) \geq C_i e^{-\gamma n}$$

for $n \geq 0$, $i = 1, \ldots, q$, ρ is the Riemannian distance. Then the attractor admits fintely many ergodic SRB measures.

A simple non-example

- Assumption that N is a finite union of submanifolds is required for arguments. Result may not hold if N has infinitely many components.
- Example: Take a countable number of horizontal lines in $(-1,1)^2$.
- In each resulting section, embed a copy of the geometric Lorenz attractor.
- Each section admits its own SRB measure.
- Granted, this example is not topologically transitive, and thus not very informative.
- Is there a transitive hyperbolic attractor with an infinite-component singular set admitting countably infinitely many ergodic SRB measures?

Idea of proof: Preliminary constructions

- Recall $K^+ = \{x \in K : f^n(x) \notin N^+ \, \forall n \geq 0\}$ and $D = \bigcap_{n \geq 0} f^n(K^+)$.
- Given $\delta > 0$, let $B_\delta^- \subset D$ consist of those $x \in D$ for which $W_\delta^u(y)$ exists and contains x, for some $y \in D$.
- Suppose $\delta_1 < \delta_2$. Then $B_{\delta_2}^- \subseteq B_{\delta_1}^-$.
  - Indeed, if $x \in B_{\delta_2}^-$, then $x \in W_{\delta_2}^u(y)$ for some $y \in D$.
  - By certain regularity hypotheses, $D \cap W_{\delta_2}^u(y)$ has full measure, so can pick $y' \in W_{\delta_2}^u(y)$ that is with δ_1-distance to x.
  - Follows that $x \in B_{\delta_1}^-$.

Lemma

Each B_δ^- admits at most finitely many ergodic SRB measures.

Proving first lemma

The proof is a Hopf argument:

- Let $\varphi \in C^0(K)$, and let $\Lambda^0 \subset \Lambda$ be the points on which the limits
$$\varphi_\pm(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi|_\Lambda \left(f^{\pm k}(x)\right)$$
both exist. (Then $\mu(\Lambda^0) = 1$ by Birkhoff ergodic theorem, w.r.t. any invariant μ.)
- Partition Λ^0 into equivalence classes on which φ_+ and φ_- are constant (and equal).
- These equivalence classes are clopen in Λ.
- Since Λ is compact, there may only be finitely many equivalence classes.
- Any ergodic SRB measure on Λ is supported on one of these equivalence classes, and each equivalence class can support at most one ergodic SRB measure.

Second lemma

- Recall if $\delta_1 < \delta_2$, then $B^-_{\delta_2} \subseteq B^-_{\delta_1}$.
- If μ_2 is an ergodic SRB measure for $B^-_{\delta_2}$, it is also one for $B^-_{\delta_1}$.
- So as $\delta \searrow 0$, B^-_δ may admit more SRB measures.
- Does $B^- = \bigcup_{\delta>0} B^-_\delta$ then admit infinitely many?

Lemma

There exists a $\delta_0 > 0$ so that if μ is an ergodic SRB measure of $f : \Lambda \to \Lambda$, $\mu(B^-_{\delta_0}) > 0$.

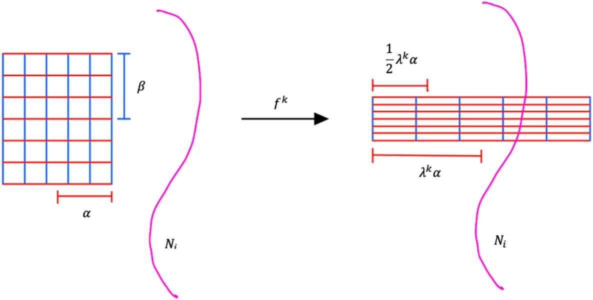
- So the answer is no: if there were infinitely many, $B^-_{\delta_0}$ would be charged by all of them, which contradicts the first lemma. This proves the main result. □

Proving second lemma

- B_δ^- is the set of points whose local unstable leaves have radius $\delta > 0$.
- Recall $N = \bigcup_{i=1}^m N_i$, and if U is a neighborhood of N, then $f(U)$ is a neighborhood of N^-.
- Since $f^j(N^-) \cap N = \varnothing$ for $1 \leq j < k$, $\lambda^k > 2$ ($\lambda > 1$ expansive constant), and N and $f^k(N^-)$ are closed, there is a radius $Q > 0$ so that:
  - the open neighborhoods $B_Q(N_i)$, $1 \leq i \leq m$, are disjoint;
  - $f^j(B_Q(N_i)) \cap N = \varnothing$ for $1 \leq j < k$.
- We let $\delta_0 < Q$.
- Choose ergodic SRB measure μ, and μ-generic point $x \in D$. Using hyperbolic product structure, construct rectangle R of stable leaves of radius $\beta > 0$ and unstable leaves of radius $\alpha > 0$. Then $\mu(R) > 0$.
- $f(R)$ is a rectangle whose unstable leaves have length $\lambda\alpha$.

Proving second lemma (cont)

- **Idea:** Use iterates of R to extend the unstable leaves until they are of radius $\geq \delta_0$. Once this happens, $f^j(R) \subset B^-_{\delta_0}$, and $\mu(B^-_{\delta_0}) > \mu(f^j(R)) = \mu(R) > 0$.
- **Obstruction:** What if $f^j(R) \cap N \neq \varnothing$ before $\lambda^j \alpha > \delta_0$?
- Since $\delta_0 > Q$, $Q =$ radius of disjoint neighborhoods of components of N, $f^j(R)$ lies in one of these neighborhoods.
- Choose new rectangle R_1 of radius $\alpha_1 \geq \frac{1}{2}\lambda^j \alpha$.



Proving second lemma (cont)

- Iterate R_1 until either:
  - $\lambda^j \alpha_1 \geq \delta_0$ (in which case $\mu(B^-_{\delta_0}) > 0$, and we're done); or
  - $f^{j_1}(R_1) \cap N \neq \varnothing$ for some $j_1 \geq k$ (since $R_1 \subset B_Q(N_i)$).
- In latter case, take new rectangle $R_2 \subset f^{j_1}(R_1) \setminus N$ with unstable leaves of radius $\alpha_2 \geq \frac{1}{2}\lambda^{j_1}\alpha_1$.
- Repeat this process. Each time a rectangle intersects N, we take a leaf of at least half the radius, creating a sequence of rectangles $\{R_\ell\}$ with unstable leaves of radii

$$\alpha_\ell \geq \frac{1}{2^\ell}\lambda^{j_1+\cdots+j_\ell}\alpha_1 > \frac{\lambda^{k\ell}}{2^\ell}\alpha_1 = \left(\frac{\lambda^k}{2}\right)^l \alpha_1,$$

  with each $j_\ell \geq k$ the time it takes for R_ℓ to intersect N.
- Since $\lambda^k > 2$, this will eventually exceed δ_0, at which point $0 < \mu(R_\ell) \leq \mu(B^-_{\delta_0})$. □

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