Hyperbolic Dynamics

Basic examples and current settings

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Introduction

Definition

- A (discrete) **dynamical system** is a pair (X, f), with X a set of points (almost always either a topological space or a measure space), and $f : X \to X$ a map (almost always either continuous or measurable, depending on the structure of X).
- Given $x \in X$, the **forward orbit** of x is the set $\mathcal{O}^+(x) = \{f^n(x) : n \in \mathbb{N}_0\}$. If f is invertible, the **full orbit** (or just the orbit) is the set $\mathcal{O}(x) = \{f^n(x) : n \in \mathbb{Z}\}$.
- A point $x \in X$ is **periodic** if $f^n(x) = x$ for some $n \geq 1$ (in which case we say x has period n, or x is n-periodic). A point $x \in X$ is **fixed** if $f(x) = x$.

Example: Angle Doubling

Let $X = \mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$, and let $f = E_2 : x \mapsto= 2x \pmod 1$.

Properties:

- "*Chaotic*": Points that are arbitrarily close together become far apart after sufficiently many iterations
- Periodic points of E_2 are dense in $\mathbb{S}^1$
- *Topologically transitive*: there is a point $x \in \mathbb{S}^1$ with a dense forward orbit.
- *Measure-preserving* with respect to Lebesgue measure: $\lambda(A) = \lambda\left(E_2^{-1}(A)\right)$.

Two orbits:

- $\mathcal{O}^+(0.11) = \{0.11, 0.22, 0.44, 0.88, 0.76, 0.52, 0.04, \ldots\}$
- $\mathcal{O}^+(0.12) = \{0.12, 0.24, 0.48, 0.96, 0.92, 0.84, 0.68, \ldots\}$

Example: 2-Shift

Let $X = \Omega_2^+ = \{0,1\}^{\mathbb{N}_0}$, and let $f = \sigma : \Omega_2^+ \to \Omega_2^+$ be defined by $\sigma(\omega)_i = \omega_{i+1}$ (ie. the sequence ω gets shifted to the left by 1, and the 0^{th} letter gets deleted).

Metric on Ω_2^+:

$$d(\omega, \omega') = 2^{-\min\{i : \omega_i \neq \omega_i'\}}$$

Open balls in Ω_2^+ are cylinders: for $\alpha_0, \ldots, \alpha_n \in \{0,1\}$:

$$C_{\alpha_0 \ldots \alpha_n} = \left\{\omega \in \Omega_2^+ : \omega_i = \alpha_i \; \forall \, 0 \leq i \leq n\right\}$$

General cylinders: for $\alpha_0, \ldots, \alpha_n \in \{0,1\}$ and nonnegative integers $j_0 < j_1 < \cdots < j_n$:

$$C_{\alpha_0 \ldots \alpha_n}^{j_0 \ldots j_n} = \{\omega \in \Omega : \omega_{j_k} = \alpha_{j_k} \; \forall \, 0 \leq k \leq n\}$$

Example: 2-shift

Orbit example:

$$\begin{aligned} \omega &= 00011100001010001111\ldots, \\ \sigma(\omega) &= 0011100001010001111\ldots, \\ \sigma^2(\omega) &= 011100001010001111\ldots, \\ \sigma^3(\omega) &= 11100001010001111\ldots, \\ \sigma^4(\omega) &= 1100001010001111\ldots, \end{aligned}$$

Easy to show σ is continuous.

Factors, conjugacies, and isomorphisms

Definition

- Suppose (X, f) and (Y, g) are topological dynamical systems, and $h : X \to Y$ is a surjective continuous map so that $h \circ f = g \circ h$. Then (Y, g) is a **topological factor** of (X, f). If h is a homeomorphism, then (X, f) and (Y, g) are **topologically conjugate**.
- Suppose (X, f) and (Y, g) are measurable dynamical systems, with X and Y finite-measure, and $h : X \to Y$ is a measurable and measure-preserving map that restricts to a bijection $X' \to Y'$, with $X' \subseteq X$ and $Y' \subseteq Y$ full-measure, so that $h \circ f = g \circ h$. Then (X, f) and (Y, g) are **measure-theoretically isomorphic** (or just isomorphic if the context is clear).

Angle doubling and 2-shift

Suppose we express the angle doubling map as doubling numbers expressed in binary:

$$\begin{aligned} 0.11 &= 0.000111000010100011111\ldots_2 \\ E_2(0.11) = 0.22 &= 0.00111000010100011111\ldots_2 \\ E_2^2(0.11) = 0.44 &= 0.0111000010100011111\ldots_2 \\ E_2^3(0.11) = 0.88 &= 0.111000010100011111\ldots_2 \\ E_2^4(0.11) = 0.76 &= 0.11000010100011111\ldots_2 \end{aligned}$$

Comparing this to the example orbit from 2 slides ago, the formal string of 0s and 1s are the same!

Angle doubling and 2-shift

Define $h : \Omega_2^+ \to \mathbb{S}^1$ sending $\omega = \omega_0\omega_1\omega_2\ldots \mapsto 0.\omega_0\omega_1\omega_2\ldots_2$. With the topology on Ω_2^+, h is surjective and continuous, and as we saw, $h \circ E_2 = \sigma \circ h$. So $\left(\mathbb{S}^1, E_2\right)$ is a factor of $\left(\Omega_2^+, \sigma\right)$.

Let $\mu : \mathcal{B}\left(\Omega_2^+\right) \to [0,1]$ be determined by

$$\mu\left(C^{j_0\cdots j_n}_{\alpha_0\ldots\alpha_n}\right) = 2^{-n}$$

Then μ is an example of a *Bernoulli probability measure* on Ω_2^+.

Let $X' \subseteq \Omega_2^+$ be the set of strings in Ω_2^+ that do not end in a tail of all 0s or all 1s. Since μ is non-atomic and X' is countable, $\mu(X') = 0$. Note $\lambda(h(X')) = 0$.

Can show h is measurable and measure-preserving, so $(\mathbb{S}^1, E_2)$ and (Ω_2^+, σ) are measure-theoretically isomorphic.

Setting of Hyperbolic Dynamics

Suppose M is an n-dimensional ($n \geq 2$) C^1 Riemannian manifold (ie. the tangent vector space at each point $x \in M$ has an inner product, and this inner product varies smoothly over M).

Now suppose $f : M \to M$ is a C^r local diffeomorphism for some $r \geq 1$ (meaning for every $x \in M$, there is an open neighborhood $U \subset M$ so that $f|_U : U \to f(U)$ is a diffeomorphism).

Hyperbolic Sets

Definition

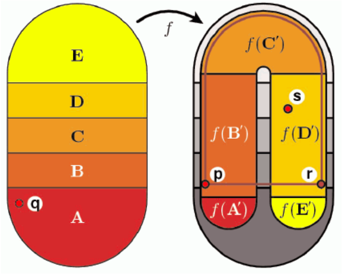
Let $U \subset M$ be open so that $f : U \to f(U)$ is a diffeomorphism. A compact f-invariant set $\Lambda \subset U$ is a **hyperbolic set** if there is a $\lambda \in (0,1)$, $C > 0$, and a splitting $T_xM = E^s(x) \oplus E^u(x)$ at each tangent plane for $x \in \Lambda$ so that:

1. $\|Df_x^n v\| \leq C\lambda^n \|v\|$ for every $v \in E^s(x)$, $n \geq 0$;
2. $\|Df_x^{-n} v\| \leq C\lambda^n \|v\|$ for every $v \in E^u(x)$, $n \geq 0$;
3. $Df_x(E^s(x)) = E^s(f(x))$ and $Df_x(E^u(x)) = E^u(f(x))$.

NOTE: Λ may not be a submanifold of M (often not locally homeomorphic to $\mathbb{R}^n$ at *any* point $x \in \Lambda$).

Smale Horseshoe

Let $S \subset \mathbb{R}^2$ be a square with two sides capped by half discs, and $f : S \to S$ a diffeomorphism onto its image, stretching S vertically, contracting horizontally, and folding in half, like so:

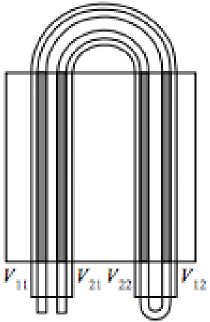


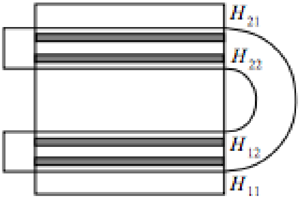
Notice only B and D have images intersecting the central square. So if a point is to remain in the square, it has to always stay inside of sets B and D.

Smale Horseshoe

Iterating the horseshoe map $f : S \to S$ forward twice more,we get a progressively more "coiled" horseshoe.

Taking preimage $f^{-1}(B)$, we get two thin horizontal rectangles: one inside B, and one inside D. Ditto $f^{-1}(D)$.

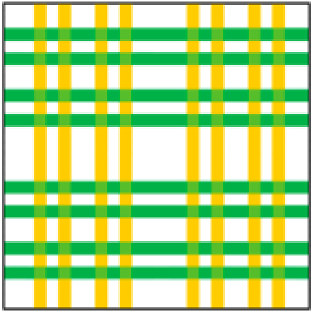




Smale Horseshoe

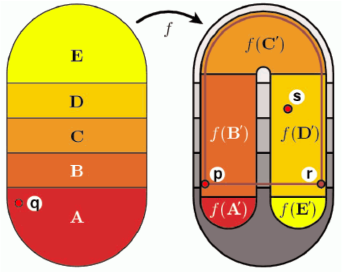
The intersection of all forward images of B and D form a Cantor set, as does the intersection of all preimages.

The resulting set $\Lambda = \bigcap_{n=-\infty}^{\infty} f^n(B) \cup f^n(D)$ is a product of Cantor sets, and a hyperbolic set in S. Note $f : \Lambda \to \Lambda$ is a bijection.



Smale Horseshoe

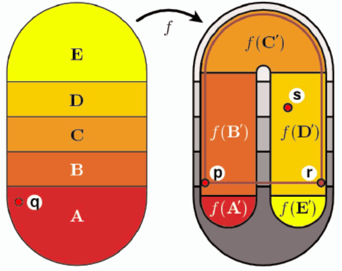
In the case of the horseshoe, the contracting directions $E^s(x)$ are horizontal lines at each point (notice if two points in Λ share a horizontal coordinate, they grow closer together), and the expanding directions $E^u(x)$ are vertical lines (if two points share a vertical coordinate, they grow closer together *in backwards time*).



Smale Horseshoe

Much like the expanding map $E_2 : \mathbb{S}^1 \to \mathbb{S}^1$, the horseshoe $f : \Lambda \to \Lambda$ can be encoded into a *symbolic* system: the *full shift* $\Omega_2 := \{0,1\}^{\mathbb{Z}}$, with map $\sigma : \Omega_2 \to \Omega_2$ given by $\sigma(\omega)_i = \omega_{i+1}$.

In this example, p has symbolic representation $\cdots 000 \cdots$, s has symbolic representation $\cdots 111 \cdots$, and r has symbolic representation $\cdots 00100 \cdots$. (p stays in B and s stays in D, but r is in D once and otherwise stays in B.)

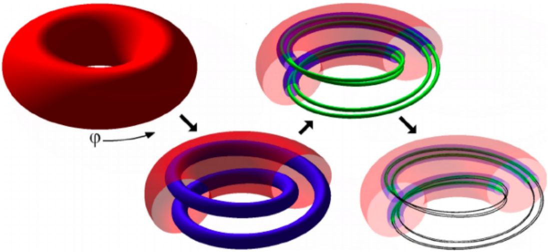


Smale Solenoid

Let $M = \mathbb{S}^1 \times \mathbb{D}^2$: the solid torus. Define the map $f : M \to M$ by

$$f(\varphi, x, y) = \left(2\varphi, \alpha x + \frac{1}{2}\cos 2\pi\varphi, \alpha y + \frac{1}{2}\sin 2\pi\varphi\right)$$

for some fixed $\alpha \in (0, 1/2)$. Then f is a diffeomorphism onto its image, a solid torus stretched by a factor of 2, contracted by a factor of α, and twisted inside the original solid torus:



Smale Solenoid

The closed invariant set $\Lambda = \bigcap_{n \geq 1} f^n(M)$ is known as the *Smale-Williams solenoid*. Note $f|_\Lambda : \Lambda \to \Lambda$ is bijective.

The solenoid is a hyperbolic set, in fact a hyperbolic *attractor* (meaning the orbit of every point $p \in M$ approaches a sequence of points in Λ, i.e. $d(f^n(p), \Lambda) \to 0$).

Locally, the solenoid is a product of a Cantor set with an open interval.

The stable subspaces $E^s(p)$ are parallel to the 2-dimensional cross-sectional discs of M.

The unstable subspaces $E^u(p)$ are along the "open intervals" in the *local product structure* of Λ.

Analyzing the solenoid

Define $\Phi = \left\{ (\varphi_n)_{n=0}^{\infty} \in \left(\mathbb{S}^1\right)^{\mathbb{N}_0} : \varphi_i = 2\varphi_{i+1} \pmod 1 \right\}$. Then Φ is a closed subgroup of the additive topological group $\left(\mathbb{S}^1\right)^{\mathbb{N}_0}$.

The map $\alpha : \Phi \to \Phi$ given by $\alpha(\varphi_0, \varphi_1, \ldots) = (2\varphi_0, \varphi_0, \varphi_1, \ldots)$ is a group automorphism and a homeomorphism.

Given $p \in \Lambda$, the first (angular) coordinates of the preimages $f^{-n}(p) = (\varphi_n, x_n, y_n)$ form a sequence $h(p) = (\varphi_n)_{n=0}^{\infty} \in \Phi$.

One can show $h : \Lambda \to \Phi$ is a homeomorphism, and $h \circ f = \alpha \circ h$. Thus (Φ, α) and (Λ, f) are topologically conjugate.

Hyperbolic Toral Automorphisms

Let $M = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 = \mathbb{R}^2/\mathbb{Z}^2$. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, and let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the action of A on $\mathbb{R}^2$.

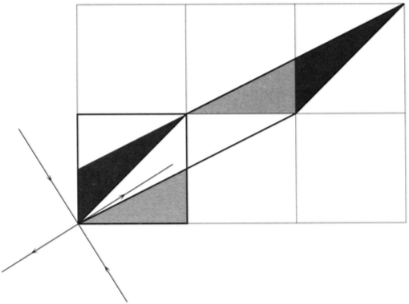
Since $\det(A) = 1$, $F\left(\mathbb{Z}^2\right) = \mathbb{Z}^2$, so F descends to a well-defined map $f : \mathbb{T}^2 \to \mathbb{T}^2$, known as a *hyperbolic toral automorphism*.

Generally, if $A \in \mathrm{SL}(n, \mathbb{Z})$ has no eigenvalues on the unit circle, then $f_A : \mathbb{T}^n \to \mathbb{T}^n$ is a hyperbolic toral automorphism.

Hyperbolic Toral Automorphisms

Eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$:

- $\lambda = \left(3 + \sqrt{5}\right)/2 > 1$, in direction of $v_\lambda := \left(\left(1 + \sqrt{5}\right)/2, 1\right)$
- $1/\lambda$, in direction of $v_{1/\lambda} := \left(\left(1 - \sqrt{5}\right)/2, 1\right)$



Hyperbolic Toral Automorphisms

Note $df_p : T_p\mathbb{T}^2 \to T_{f(p)}\mathbb{T}^2$ has matrix expression $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.

Identify $T_p\mathbb{T}^2$ with $\mathbb{R}^2$ at every $p \in \mathbb{T}^2$; then $E^s(p)$ and $E^u(p)$ are the eigenspaces spanned by $v_{1/\lambda}$ and v_λ respectively.

Thus *all of* $\mathbb{T}^2$ *is a hyperbolic set*.

If $f : M \to M$ is a diffeomorphism of a Riemannian manifold for which all of M is hyperbolic, then f is known as an *Anosov diffeomorphism*. Hyperbolic toral automorphisms are examples of Anosov diffeomorphisms.

Adapted Metrics

Recall definition of a hyperbolic set $\Lambda \subset M$:

Definition

Let $U \subset M$ be open so that $f : U \to f(U)$ is a diffeomorphism. A compact f-invariant set $\Lambda \subset U$ is a **hyperbolic set** if there is a $\lambda \in (0,1)$, $C > 0$, and a splitting $T_xM = E^s(x) \oplus E^u(x)$ at each tangent plane for $x \in \Lambda$ so that:

1. $\|Df_x^n v\| \leq C\lambda^n \|v\|$ for every $v \in E^s(x)$, $n \geq 0$;
2. $\|Df_x^{-n} v\| \leq C\lambda^n \|v\|$ for every $v \in E^u(x)$, $n \geq 0$;
3. $Df_x\left(E^s(x)\right) = E^s(f(x))$ and $Df_x\left(E^u(x)\right) = E^u(f(x))$.

Adapted Metrics

Theorem

If Λ is a hyperbolic set of $f : M \to M$ with constants C and λ, then for every $\varepsilon > 0$ there is a C^1 Riemannian metric $\langle \cdot, \cdot \rangle'$ in a neighborhood of Λ, called the adapted metric or Lyapunov metric, with respect to which f is hyperbolic and satisfies the conditions of hyperbolicity with $C' = 1$, $\lambda' = \lambda + \varepsilon$, and the subspaces $E^s(x)$ and $E^u(x)$ are ε-orthogonal. That is, $\langle v^s, v^u \rangle' < \varepsilon$ for all unit vectors $v^s \in E^s(x)$, $v^u \in E^u(x)$, and all $x \in \Lambda$.

Invariant cones and neighborhoods of Λ

Given $\varepsilon > 0$, define the sets

$$\Lambda_\varepsilon^s = \{x \in U : \mathrm{dist}\,(f^n(x), \Lambda) < \varepsilon \,\forall\, n \in \mathbb{N}_0\}\,,$$

$$\Lambda_\varepsilon^u = \left\{x \in U : \mathrm{dist}\,\left(f^{-n}(x), \Lambda\right) < \varepsilon \,\forall\, n \in \mathbb{N}_0\right\}$$

Note $E^s(x)$ and $E^u(x)$ vary continuously, so can be extended to a neighborhood $U \supset \Lambda$, so $T_x U = \widetilde{E}^s(x) \oplus \widetilde{E}^u(x)$ for every $x \in U$.

Given $x \in U$, $v \in T_x M$, suppose $v = v^s + v^u$, $v^s \in \widetilde{E}^s(x)$, $v^u \in \widetilde{E}^u(x)$. Define the *invariant stable and unstable cones of size* $\alpha > 0$:

$$K_\alpha^s(x) = \{v \in T_x M : \|v^u\| \le \alpha \|v^s\|\}\,,$$
$$K_\alpha^u(x) = \{v \in T_x M : \|v^s\| \le \alpha \|v^u\|\}\,.$$

Local Stable and Unstable Submanifolds

Theorem (Stable/Unstable Manifolds)

Let $f : M \to M$ be a C^1 diffeomorphism of a differentiable manifold and let $\Lambda \subset M$ be a hyperbolic set of f with constant f. Assume M has a Lyapunov metric for f. Then there are $\varepsilon > 0$, $\delta > 0$ such that for every $x^s \in \Lambda_\delta^s$ and every $x^u \in \Lambda_\delta^u$,

- *the sets (known as lccal unstable and local stable manifolds)*

$$W_\varepsilon^u(x^u) = \left\{y \in M : \operatorname{dist}\left(f^{-n}(x^s), f^{-n}(y)\right) < \varepsilon \ \forall\ n \in \mathbb{N}_0\right\},$$
$$W_\varepsilon^s(x^s) = \{y \in M : \operatorname{dist}(f^n(x^s), f^n(y)) < \varepsilon \ \forall\ n \in \mathbb{N}_0\}$$

  are C^1 embedded discs;
- $T_{y^{u/s}} W_\varepsilon^{u/s}(x^{u/s}) = E^{u/s}(x^{u/s})$ *for every* $y^{u/s} \in W_\varepsilon^{u/s}(x^{u/s})$;

Local Stable and Unstable Submanifolds

Theorem (Stable/Unstable Manifolds) (continued)

- $f\left(W_\varepsilon^s(x^s)\right) \subset W_\varepsilon^s\left(f(x^s)\right)$ *and* $f^{-1}\left(W_\varepsilon^u(f(x^u))\right) \subset W_\varepsilon^u\left(x^u\right)$;
- *if* $y^s, z^s \in W_\varepsilon^s(x^s)$, *then* $d^s\left(f(y^s), f(z^s)\right) < \lambda d^s\left(y^s, z^s\right)$, *where* d^s *is the distance along* $W_\varepsilon^s(x^s)$;
- *if* $y^u, z^u \in W_\varepsilon^u(x^u)$, *then* $d^u\left(f^{-1}(y^u), f^{-1}(z^u)\right) < \lambda d^u\left(y^u, z^u\right)$, *where* d^u *is the distance along* $W_\varepsilon^u(x^u)$;
- *if* $0 < \mathrm{dist}(x^s, y) < \varepsilon$ *and* $\exp_{x^s}^{-1}(y) \in K_\delta^u(x^s)$, *then* $\mathrm{dist}\left(f(x^s), f(y)\right) > \lambda^{-1}\mathrm{dist}\left(x^s, y\right)$;
- *if* $0 < \mathrm{dist}(x^u, y) < \varepsilon$ *and* $\exp_{x^u}^{-1}(y) \in K_\delta^s(x^u)$, *then* $\mathrm{dist}\left(f(x^u), f(y)\right) < \lambda \mathrm{dist}\left(x^s, y\right)$;
- *if* $y^s \in W_\varepsilon^s\left(x^s\right)$, *then* $W_\alpha^s\left(y^s\right) \subset W_\varepsilon^s\left(x^s\right)$ *for some* $\alpha > 0$, *and if* $y^u \in W_\varepsilon^u\left(x^u\right)$, *then* $W_\beta^u\left(y^u\right) \subset W_\varepsilon^u\left(x^u\right)$ *for some* $\beta > 0$.

Local Maximality and Local Product Structure

Definition

- A hyperbolic set $\Lambda \subset M$ of $f : U \to M$ is *locally maximal* if there is an open set V such that $\Lambda \subset V \subset U$ and $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(V)$.
- Λ has *local product structure* if there are sufficiently small $\varepsilon > 0$ and $\delta > 0$ such that:
  1. for all $x, y \in \Lambda$, $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ consists of at most one point, which belongs to Λ; and,
  2. for $x, y \in \Lambda$ with $d(x, y) < \delta$, the intersection consists of exactly one point $[x, y] = W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$, and the intersection is transverse.

Local Maximality and Local Product Structure

If Λ has local product structure, then there is a neighborhood $U(x)$ of every $x \in \Lambda$ so that

$$U(x) \cap \Lambda = \{[y, z] : y \in U(x) \cap W_\varepsilon^s(x),\ z \in U(x) \cap W_\varepsilon^u(x)\}.$$

Theorem

A hyperbolic set Λ is locally maximal if and only if it has a local product structure.

Global Stable and Unstable Submanifolds

Global analogue of stable/unstable submanifolds for points $x \in \Lambda$:

$$W^s(x) := \{y \in M : d\left(f^n(x), f^n(y)\right) \to 0 \text{ as } n \to \infty\},$$
$$W^u(x) := \left\{y \in M : d\left(f^{-n}(x), f^{-n}(y)\right) \to 0 \text{ as } n \to \infty\right\}.$$

Theorem

There is an $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, for every $x \in \Lambda$,

$$W^s(x) = \bigcup_{n=0}^{\infty} f^{-n}\left(W^s_\varepsilon\left(f^n(x)\right)\right), \quad \textit{and}$$
$$W^u(x) = \bigcup_{n=0}^{\infty} f^{n}\left(W^u_\varepsilon\left(f^{-n}(x)\right)\right).$$

Anosov Diffeomorphisms

An *Anosov diffeomorphism* is a diffeomorphism $f : M \to M$ of a connected differentiable manifold for which M is a hyperbolic set.

Suppose N is a simply-connected nilpotent Lie group, Γ a uniform discrete subgroup of N. Then $M := N/\Gamma$ is a *nilmanifold*.

If $\overline{f} : N \to N$ is an automorphism of N that preserves Γ and whose derivative at the identity is hyperbolic, then the induced map $f : M \to M$ is Anosov.

Conjecture: Up to finite coverings, all Anosov diffeomorphisms are topologically conjugate to automorphisms of nilmanifolds.

Anosov Diffeomorphisms

The global stable and unstable manifolds $W^s(x)$ and $W^u(x)$ of an Anosov diffeomorphisms form stable and unstable *foliations* of the manifold M.

For $M = \mathbb{T}^2$, $f : \mathbb{T}^2 \to \mathbb{T}^2$ generated by linear hyperbolic map $A = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$, the unstable leaves of the foliation (i.e. the global stable submanifolds) are curves parallel to the eigendirections of $\lambda = \left(3 + \sqrt{5}\right)/2$. Stable leaves are curves parallel to the eigendirections of $1/\lambda$.

Anosov Diffeomorphisms

A point $x \in M$ is *nonwandering* if for every neighborhood $U \ni x$ there is an $n \geq 1$ so that $f^n(U) \cap U \neq \varnothing$. The set of all nonwandering points is denoted $NW(f)$.

A diffeomorphism $f \in \mathrm{Diff}^1(M)$ is *structurally stable* if for every $\varepsilon > 0$, there is a neighborhood $\mathcal{U} \subset \mathrm{Diff}^1(M)$ of f such that for every $g \in \mathcal{U}$ there is a homeomorphism $h : M \to M$ with $h \circ f = g \circ h$ and $d_0(h, \mathrm{Id}) < \varepsilon$.

Properties of Anosov diffeomorphisms:

- Anosov diffeomorphisms form an open (possibly empty) subset of $\mathrm{Diff}^1(M)$.
- Anosov diffeomorphisms are structurally stable.
- The set of periodic points is dense in $NW(f)$.

Anosov Diffeomorphisms

Theorem

Let $f : M \to M$ be an Anosov diffeomorphism. The following are equivalent:

- $NW(f) = M$;
- *every unstable manifold is dense in M;*
- *every stable manifold is dense in M;*
- *f is topologically transitive (i.e. there exists a dense orbit);*
- *f is topologically mixing (i.e. for every $U, V \subset M$, there is $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \varnothing$ for $n \geq N$).*

Conjecture: These statements hold for *every* Anosov diffeomorphism.

Markov Partitions

Anosov diffeomorphisms are often encoded into a symbolic system via a *Markov partition*.

Definition

A **Markov partition** $\mathcal{P}$ of a manifold M for an invariant subset Λ of a diffeomorphism $f : M \to M$ is a (typically finite) collection of subsets $R_i \subset M$, called *rectangles*, such that for all i, j, k:

- $R_i = \overline{\mathrm{int} R_i}$;
- $\mathrm{int} R_i \cap \mathrm{int} R_j = \varnothing$ if $i \neq j$;
- if $f^m(\mathrm{int} R_i) \cap \mathrm{int} R_j \cap \Lambda = \varnothing$ for some $m \in \mathbb{Z}$, and $f^n(\mathrm{int} R_j) \cap \mathrm{int} R_k \cap \Lambda \neq \varnothing$ for some $n \in \mathbb{Z}$, then $f^{m+n}(\mathrm{int} R_i) \cap \mathrm{int} R_k \cap \Lambda \neq \varnothing$.

Markov Partitions

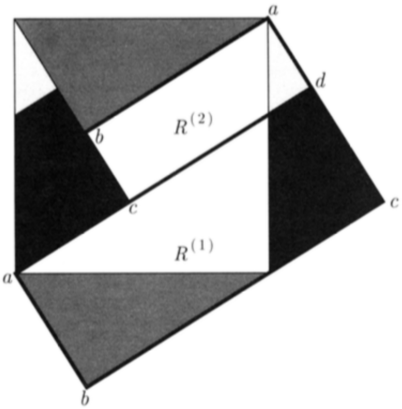
The set of two-sided sequences of the alphabet $\{R_i\}$ gives a symbolic dynamical system, whose orbits correspond to the orbits of $f : M \to M$

For $M = \mathbb{S}^1$, $f = E_2$, even though f is not hyperbolic, the partition $R_0 = [0, 1/2]$, $R_1 = [1/2, 1]$ is a Markov partition: the binary-expanded point $0.0001110000101..._2 \in \mathbb{S}^1$ gets sent first to R_0 in the first 3 iterations of E_2, then R_1 for the next three iterations, then R_0 for the next four, etc.

Theorem

Every Anosov diffeomorphism admits a Markov partition.

Markov Partitions



Thank You!

References:

- Katok, A. & Hasselblatt, B. (1995) *Introduction to the Modern Theory of Dynamical Systems*. Cambridge: Cambridge University Press.
- Brin, M. & Stuck, G. (2015) *Introduction to Dynamical Systems*. Cambridge: Cambridge University Press.