Thermodynamics of Pseudo-Anosov Diffeomorphisms

Warwick University Ergodic Theory and Dynamical Systems Seminar

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Why study these?

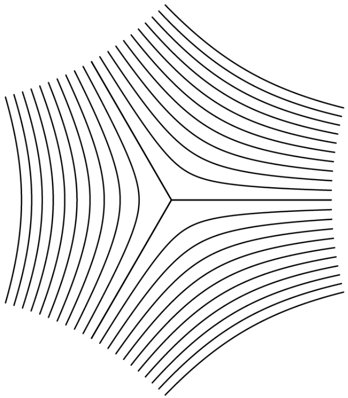
- A diffeomorphism $f : M \to M$ of a Riemannian manifold is **Anosov** if at every $x \in M$, the tangent space splits into two invariant subspaces $T_xM = E^u(x) \oplus E^s(x)$:

$$\begin{aligned} |df_x^n v| &\leq C\lambda^{-n}|v| \quad \text{for every } v \in E^s(x); \quad \text{and} \\ \left|df_x^{-n} v\right| &\leq C\lambda^{-n}|v| \quad \text{for every } v \in E^u(x). \end{aligned}$$

  for some $\lambda > 1$ and $C > 0$.
- In low dimensions, these are hyperbolic toral autmorphisms, $A : \mathbb{T}^2 \to \mathbb{T}^2$, $A \in \mathrm{SL}(2, \mathbb{Z})$, and their perturbations.
- If we can't have Anosov maps on other surfaces, what's the next best thing?

Foliations with singularities

- A *foliation with singularities* $\mathcal{F}$ of a 2-manifold M, for our purposes, is a foliation of M where there are finitely many points $x_1, \ldots, x_l \in M$ at which some number $p = p(x_k) = p_k \geq 3$ of the leaves meet (these are the *prongs* of the singularity):

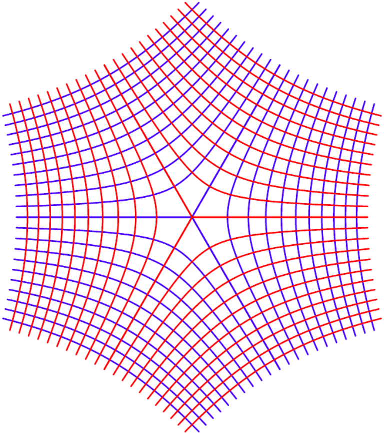


Pseudo-Anosov Homeomorphisms

- A homeomorphism $f : M \to M$ of a 2-manifold M is *pseudo-Anosov* if there are two f-invariant foliations with singularities, $\mathcal{F}^s$ and $\mathcal{F}^u$, for which:
  1. the foliations share the same singularities, and the same number of prongs;
  2. the foliations intersect transversally away from the singularities;
  3. there is a $\lambda > 1$ such that for x, y in the same $\mathcal{F}^s$-leaf, $\rho^s(f(x), f(y)) = \lambda^{-1}\rho^s(x, y)$, and for x, y in the same $\mathcal{F}^u$-leaf, $\rho^u(f(x), f(y)) = \lambda\rho^u(x, y)$;

  where ρ^s and ρ^u are the distances in the $\mathcal{F}^s$ and $\mathcal{F}^u$ foliations with respect to a Riemannian metric on M that has a density vanishing at the singularities.

Pseudo-Anosov Homeomorphisms



The blue curves represent the stable foliation, along which nearby points contract; and the red curves represent the unstable foliation, along which nearby points expand.

Nielsen-Thurston Classification

Theorem (Nielsen, Thurston)

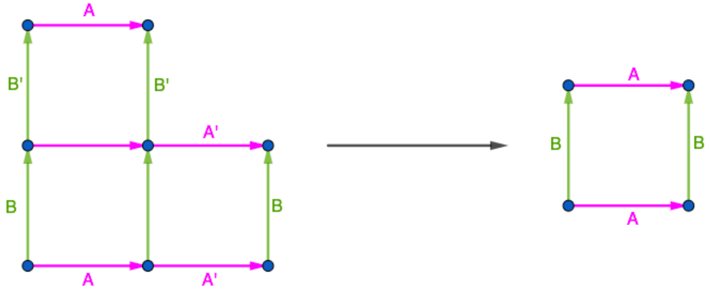
Any homeomorphism on a compact topological manifold M is isotopic to a map f that is one of the following:

- *f is* **periodic***: there is a positive integer m with $f^m = \mathrm{Id}$;*
  - *EG. $f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$;*
- *f is* **reducible***: there is a closed curve on M that is f-invariant (these are also known as Dehn twists);*
  - *EG. $f = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$;*
- *f is* **pseudo-Anosov;**
  - *EG. $f = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} : \mathbb{T}^2 \to \mathbb{T}^2$.*

The pseudo-Anosov maps form an open set in the homeomorphisms of M, and exhibit the most interesting dynamical properties.

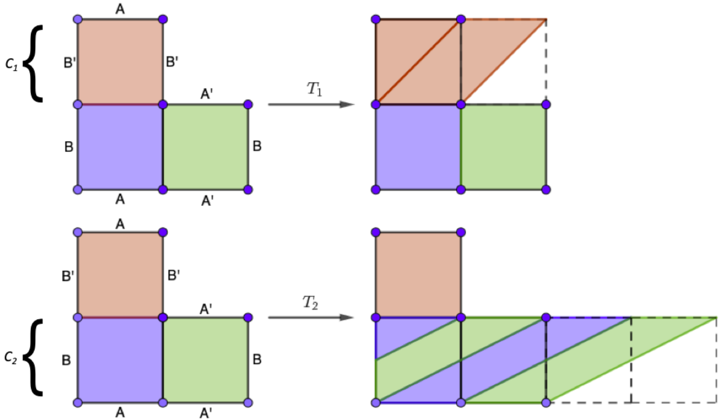
Examples from Anosov maps

- An Anosov diffeomorphism is a pseudo-Anosov homeomorphism with no singularities.
- Linear Anosov maps on $\mathbb{T}^2$ lift to pseudo-Anosov maps on higher-genus surfaces via branched coverings (may be necessary to lift powers of Anosov maps).



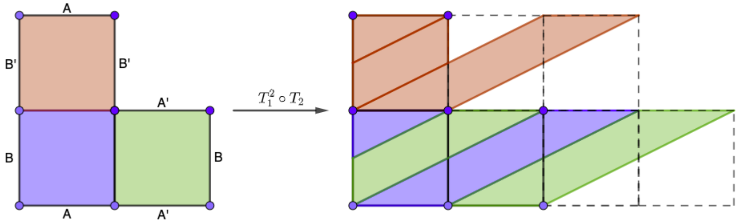
Example from Dehn twists

- Orientable genus-2 surface S_2 can be horizontally split into two cylinders C_1 and C_2, each of which admits a Dehn twist T_1 and T_2 resp. Note $dT_1 = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ on C_1, and $dT_2 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ on C_2, away from the identified vertex (singularity).



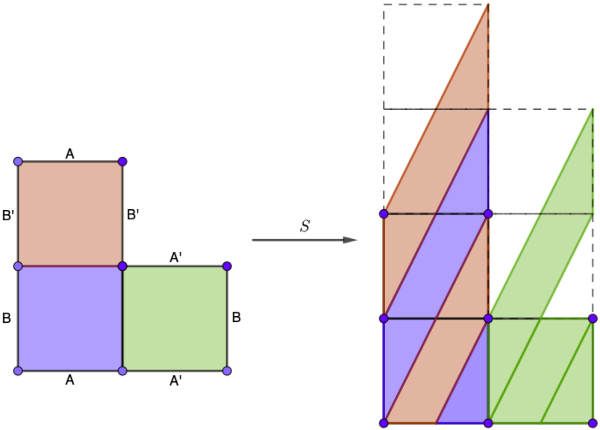
Example from Dehn twists

- The Dehn twists T_1 and T_2 can be composed to form a horizontal "multi-twist" $T := T_1^2 \circ T_2$, whose differential away from the vertex is $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$:



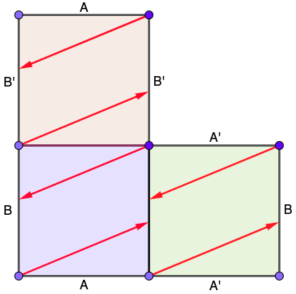
Example from Dehn twists

- A similar vertical multi-twist S can be defined on the cylinder made from the red and blue squares, and the cylinder made from the green square. The corresponding differential is $\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$.



Example from Dehn twists

- $T \circ S$ has the constant differential $\left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 5 & 2 \\ 2 & 1 \end{smallmatrix}\right)$.
- Eigenvalues are $3 - 2\sqrt{2}$ and $3 + 2\sqrt{2}$, with resp. eigenvectors $\left(\begin{smallmatrix} 1-\sqrt{2} \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1+\sqrt{2} \\ 1 \end{smallmatrix}\right)$.
- This is a pseudo-Anosov map whose stable/unstable foliations are parallel to the eigendirections for $3 - 2\sqrt{2}$ and $3 + 2\sqrt{2}$, resp. The vertex is a 6-pronged singularity (unstable prongs illustrated).



- In fact, this map is the lift of the linear Anosov map on $\mathbb{T}^2$ induced by $\left(\begin{smallmatrix} 5 & 2 \\ 2 & 1 \end{smallmatrix}\right)$.

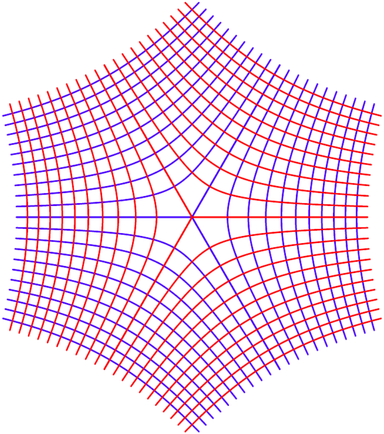
Properties

Suppose $f : M \to M$ is a pseudo-Anosov homeomorphism with expansion factor $\lambda > 1$.

- There is a Riemannian metric on M inducing a volume ν under which f is invariant, given locally by the product of the stable/unstable lengths ν^s and ν^u in $\mathcal{F}^s$ and $\mathcal{F}^u$.
- If U is a neighborhood of a singularity $x_i \in M$ and $\varphi : U \to \mathbb{R}^2$ is a coordinate chart, ν has a density with respect to $\varphi_*^{-1}(\mathrm{Leb})$ vanishing at x_i.
- If $x \in M$ is not a singularity, then f is smooth at x and there are orthonormal bases of $T_x M$ and $T_{f(x)} M$ with respect to which df_x has the matrix form $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$.

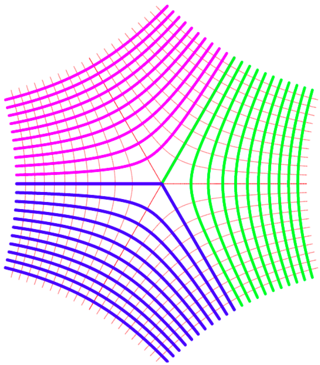
Behavior at Singularities

- The orthonormal basis are tangent vectors of the stable and unstable leaves. Along different prongs of the singularities, matrix form of df approaches different rotations of $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$. In particular, f is not differentiable at the singularities.



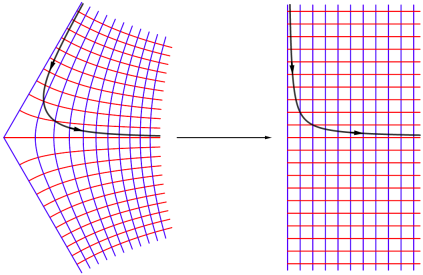
Simplifying assumptions

- Since f is a homeomorphism, f permutes the singularities, so we may assume singularities are fixed points. Furthermore, we may assume that near singularities, the open sectors between the stable prongs are invariant under f.



Slow-down procedure

- Each open sector is homeomorphic to the right half-plane, where in coordinates we have $s_1 = \mathrm{const}$ are the stable leaves, and $s_2 = \mathrm{const}$ are unstable leaves.



- In these coordinates, the map has the form

$$f(s_1, s_2) = \left(\lambda s_1,\ \lambda^{-1} s_2\right) \tag{1}$$

which is the time-1 map of the flow given by

$$\dot{s}_1 = s_1 \log \lambda, \quad \dot{s}_2 = -s_2 \log \lambda.$$

Slow-down procedure

- For each singularity x_i, choose coordinate ball of radius $r_i > 0$ in which $f(s_1, s_2) = (\lambda s_1, \lambda^{-1} s_2)$ in each sector.
- Let $0 < \tilde{r}_i < r_i$. Suppose x_i has p prongs. Define a "slow-down" function $\Psi_p : [0, \infty) \to \mathbb{R}$ so that:
  1. For $u \leq \tilde{r}_i^2$, we have
  $$\Psi_p(u) = C_p u^{(p-2)/p},$$
  where $C_p = (p/2)^{(2p-4)/p}$;
  2. Ψ_p is C^∞ except at 0;
  3. $\dot{\Psi}_p(u) \geq 0$ for $u > 0$;
  4. $\Psi_p(u) = 1$ for $u \geq r_i^2$.
- Let G_p be the time-1 map of the flow given by:
$$\begin{cases} \dot{s}_1 = (\log \lambda) s_1 \Psi_p\left(s_1^2 + s_2^2\right), \\ \dot{s}_2 = -(\log \lambda) s_2 \Psi_p\left(s_1^2 + s_2^2\right). \end{cases} \tag{2}$$
- Same trajectories as $\dot{s}_1 = (\log \lambda) s_1$, $\dot{s}_2 = -(\log \lambda) s_2$, but slower.

Slow-down procedure

- In coordinates, for $|(s_1, s_2)| \geq r_i$, we have $G_p(s_1, s_2) = f(s_1, s_2)$, so we define $g : M \to M$ in coordinates by

$$g(x) = \begin{cases} G_p(s_1, s_2) & \text{if } x = (s_1, s_2) \text{ is near a singularity,} \\ f(x) & \text{otherwise.} \end{cases}$$

- Compare to the *Katok map* $G : \mathbb{T}^2 \to \mathbb{T}^2$, which is a toral automorphism that has similarly been slowed down at the origin.
  - After slow-down, the Katok map is conjugated with a homeomorphism to make the map Lebesgue-preserving ("blows up" trajectories near the origin).
- For smooth pseudo-Anosov maps, we instead show g preserves the measure $\Psi_p(s_1^2 + s_2^2)^{-1} ds_1 \wedge ds_2$.

Pseudo-Anosov Diffeomorphisms

Theorem (Gerber and Katok, 1982)

- *The map g is a C^∞ nonuniformly hyperbolic diffeomorphism of M.*
- *g is topologically conjugate to the pseudo-Anosov map f via a homeomorphism that is isotopic to the identity.*
- *This conjugacy is a homeomorphism only, and cannot be made C^1.*
- *In every neighborhood of the singularities, g is real analytic. Furthermore, g is Bernoulli with respect to an invariant measure given by a smooth positive density.*

Physical measures

- If we're studying ergodic theory, what measures are we interested in?
- Primarily we want to study the *physical measures*:

$$\mu\left\{x : \frac{1}{n}\sum_{k=0}^{n-1}\left(\varphi\circ f^k\right)(x) \to \int \varphi\, d\mu \quad \forall \varphi \in C^0\right\} > 0$$

- For hyperbolic systems $f : M \to M$, one important physical measures are the **Sinai-Ruelle-Bowen (SRB)** measures, which are f-invariant probability measures μ for which:
  - μ has positive Lyapunov exponents almost everywhere, and
  - μ has absolutely continuous conditional measures on unstable manifolds (w.r.t. Riemannian leaf volume).
- For Anosov and pseudo-Anosov maps that preserve area, SRB measures are simply this area.

Equilibrium states and geometric potentials

- Let $\varphi : M \to \mathbb{R}$ be continuous. A probability measure μ_φ is an **equilibrium measure** for φ if

$$P_g(\varphi) = h_{\mu_\varphi}(g) + \int_M \varphi \, d\mu_\varphi,$$

  where $h_{\mu_\varphi}(g)$ is the metric entropy of g and $P_g(\varphi)$ is the topological pressure of φ:

$$P_g(\varphi) = \sup_{\mu \in \mathcal{M}(g)} \left\{ h_\mu(g) + \int_M \varphi \, d\mu \right\}$$

- We consider equilibrium states of the *geometric t-potential*

$$\varphi_t(x) = -t \log \left| dg|_{E^u(x)} \right|.$$

  We denote $\mu_t := \mu_{\varphi_t}$.
- Observe that μ_0 is a measure of maximal entropy.

Ergodic theory of Anosov maps

- The special case $t = 1$ gives the *geometric potential* $\varphi_1(x) = -\log\det\left|dg|_{E^u(x)}\right|$, for which the equilibrium state μ_1 is SRB:

Theorem

If $f : M \to M$ is an Anosov (or more generally Axiom A) diffeomorphism, there exists a unique SRB measure for f.

Proof.

From Bowen's notes:

- Any Hölder continuous potential $\varphi : M \to \mathbb{R}$ has a unique equilibrium state μ_φ for φ.
- The geometric potential $\varphi_1(x)$ is Hölder, so $\mu_{\varphi_1} = \mu_1$ is the unique equilibrium state.
- This equilibrium state satisfies all of the requisite properties of SRB measures.

□

The pseudo-Anosov case

- Bowen's proof of this result relies on two key points:
  - The map admits a finite Markov partition.
  - Geometric potential φ_1 is Hölder.

Proposition (Fathi, Shub '79; Katok, Gerber '82)

The pseudo-Anosov homeomorphism $f : M \to M$ admits a finite Markov partition, with respect to which f is Bernoulli.

- For "linear" pseudo-Anosov homeomorphisms, this argument applies basically verbatim.
- For the slowed-down Katok-Gerber diffeomorphisms, φ_1 is not Hölder (in particular its induced map on the finite symbolic space is not Hölder).

Decay of correlations and CLT

- f has **exponential decay of correlations** with respect to a measure μ and a class of functions $\mathcal{H}$ on M if there exists $\kappa \in (0,1)$ s.t. for any $h_1, h_2 \in \mathcal{H}$,

$$\left| \int (h_1 \circ f^n)\, h_2 \, d\mu - \int h_1 \, d\mu \int h_2 \, d\mu \right| \leq C\kappa^n$$

for some $C = C(h_1, h_2) > 0$.

- f satisfies the **Central Limit Theorem** (CLT) if for any $h \in \mathcal{H}$ s.t. $h \neq h' \circ f - h'$, $h' \in \mathcal{H}$, there is $\sigma > 0$ s.t.

$$\lim_{n\to\infty} \mu\left\{ \sqrt{n}\left(\frac{1}{n} S_n(h) - \mathbb{E}(h) \right) < t \right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\tau^2/2\sigma^2}\, d\tau$$

where $S_n(h) = \sum_{i=0}^{n-1} h(f^i(x))$ and $\mathbb{E}(h) = \int_M h \, d\mu$.

Main Result

Theorem (V. 2022)

Let $g : M \to M$ be a pseudo-Anosov diffeomorphism of a compact orientable manifold M (as in the preceding construction).

1. *For any $t_0 < 0$, we may choose radii $r_i > 0$ in the construction of g s.t. for $t \in (t_0, 1)$, there is a unique equilibrium measure μ_t for the geometric potential φ_t. Further:*
    - *μ_t satisfies CLT with respect to a class of functions containing all Hölder functions;*
    - *μ_t has exponential decay of correlations with respect to this class of functions, and is hence mixing;*
    - *the map is Bernoulli with respect to μ_t;*
    - *the pressure function $t \mapsto P_g(\varphi_t)$ is real-analytic on $(t_0, 1)$.*

Main Result (cont)

2. *For $t = 1$, there are two classes of equilibrium measures associated to φ_1:*
    - *convex combinations of the Dirac measures δ_{x_i} centered at the singularities, and*
    - *a unique invariant SRB measure.*
3. *For $t > 1$, all equilibrium measures for φ_t are convex combinations of the measures δ_{x_i}.*

This result closely mirrors a similar result (Pesin, Senti, and Zhang, 2017) about the Katok map $G : \mathbb{T}^2 \to \mathbb{T}^2$.

- Replace "convex combinations of δ_{x_i}" with "the Dirac measure at the origin".

Further results

Theorem (Wang 2019)

The Katok map G has a unique equilibrium state for φ_t, for every $t < 1$, with respect to which G has the CLT and exponential decay of correlations and large deviations.

Theorem (Pesin, Senti, Shahidi 2020)

Via a gluing procedure with the Katok map, any surface admits a $C^{1+\varepsilon}$ diffeomorphism with nonzero Lyapunov exponents and polynomial decay of correlations:

$$\left| \int (h_1 \circ f^n)\, h_2 \, d\mu - \int h_1 \, d\mu \int h_2 \, d\mu \right| \leq C n^{-\kappa}$$

Question: What about pseudo-Anosovs?

- Both of these results assume the exponent $\alpha > 0$ in the slowing down of the Katok map is $< 1/2$. Our exponent is $(p-2)/p > 1/2$ when $p \geq 5$.

Young diffeomorphisms (general idea)

- The proof of the main result relies on the technology of *Young towers*.
- Given $g : M \to M$ and $\Lambda \subset M$, let $\tau : \Lambda \to \mathbb{N}$ be an *inducing time* (often first-return time) and let $G = g^\tau : \Lambda \to \Lambda$ be the *induced map*, defined by $G(x) = g^{\tau(x)}(x)$.
- The map $g : M \to M$ is a *Young diffeomorphism* with base $\Lambda \subset M$ if Λ has hyperbolic product structure, and G satisfies certain "nice" properties, including:
  - Stable (resp. unstable) leaves are invariant under G (resp. G^{-1});
  - G (resp. G^{-1}) contracts points in the same stable (resp. unstable) leaf as $n \to \infty$ (resp. $n \to -\infty$);
  - τ is integrable on some unstable leaf;
  - Distortion estimates are bounded (more on this later).

Thermodynamics of Young's diffeomorphisms

- Let $g : M \to M$ be a $C^{1+\varepsilon}$ Young diffeomorphism of a compact Riemannian manifold M with base $\Lambda \subset M$ and first return time $\tau : \Lambda \to \mathbb{N}$. Under certain arithmetic and combinatorial conditions:

Theorem (Pesin, Senti, Zhang 2016)

- *$\exists$ an equilibrium measure μ_1 for the potential φ_1, which is the unique SRB measure;*
- *$\exists$ $t_0 < 0$ s.t. for $t \in (t_0, 1)$, there is a unique equilibrium measure μ_t for φ_t on $Y := \left\{g^k(x) : x \in \Lambda, 0 \leq k \leq \tau(x) - 1\right\}$;*
- *For $t \in (t_0, 1)$, the measure μ_t has exponential decay of correlations and the CLT with respect to a class of functions $\mathcal{H}$ containing all Hölder functions on M.*

Theorem (Shahidi, Zelerowicz 2018)

If $g : M \to M$ is mixing, then (M, g, μ_t) is Bernoulli.

Constructing Tower

- Let $\mathcal{P}$ be a Markov partition for g, and let $P \in \mathcal{P}$ be a rectangle that does not contain any singularity.
- Let $\tau(x)$ be first return time of x to P.
- For $x \in P$, let $\gamma^s(x)$ and $\gamma^u(x)$ be the connected component of the intersection of the stable and unstable leaves with P.
- For x with $\tau(x) < \infty$, let $U^u(x) \subseteq \gamma^u(x)$ be an open interval containing x, and

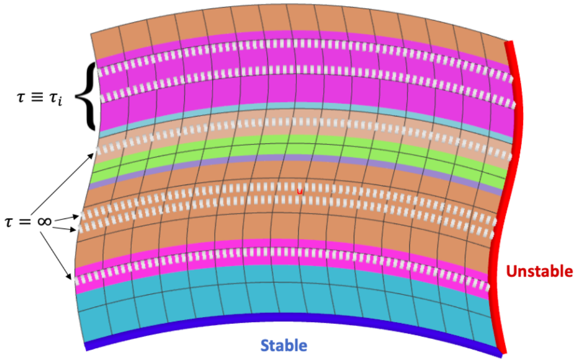
$$A^u(x) = \{y \in U^u(x) : y \in \partial P \text{ or } \tau(y) = \infty\}.$$

  Assume $U^u(x)$ is small enough s.t. $\tau|_{U^u(x) \setminus A^u(x)} \equiv \text{const } \forall x \in P$ w/ $\tau(x) < \infty$.
- Define the "stable strips":

$$\Lambda^s(x) = \bigcup_{y \in U^u(x) \setminus A^u(x)} \gamma^s(y).$$

Constructing Tower

- We get countable collection $\{\Lambda_i^s\}_{i\geq 1}$ w/ $\tau|_{\Lambda_i^s} \equiv \tau_i \in \mathbb{N}$. Define $\Lambda = \bigcup_{i\geq 1} \Lambda_i^s$.



Theorem (V. 2022)

The smooth pseudo-Anosov diffeomorphism $g : M \to M$ is a Young's diffeomorphism with tower base Λ.

Bounded Distortion

- Most properties of Young diffeomorphisms are easy to verify, and follow from corresponding properties of pseudo-Anosov diffeomorphisms. The one tricky property is *bounded distortion:*

Lemma

There exist $c > 0$ *and* $\kappa \in (0,1)$ *such that for all* $n \geq 0$, $x \in \Lambda$ *and* $y \in \gamma^s(x)$*, we have*

$$\left| \log \frac{\left| dG|_{E^u(G^n(x))} \right|}{\left| dG|_{E^u(G^n(y))} \right|} \right| \leq c\kappa^n.$$

- This bound is easy to show outside of slow-down neighborhoods. Inside the slow-down, there is a bound on how far apart $\log \left| dg|_{E^u(g^n(x))} \right|$ and $\log \left| dg|_{E^u(g^n(y))} \right|$ can be. (This is why we assume stable sectors are locally invariant.)

Equilibrium state existence: $t < 1$

- Using previous results, this gives us a unique equilibrium measure μ_t for $t < 1$ on the set

$$Y := \left\{ g^k(x) : x \in \Lambda, 0 \leq k \leq \tau(x) - 1 \right\}$$

- If $\hat{P}$ is another element of the Markov partition for (M, g), same argument gives us unique equilibrium measure $\hat{\mu}_t$ for $t < 1$ and corresponding set $\hat{Y}$.
- Assuming (M, g) is topologically transitive, since $\mu_t(U) > 0$ and $\hat{\mu}_t(\hat{U}) > 0$ for every open $U \supset P$, $\hat{U} \supset \hat{P}$, and $g^k(U) \cap \hat{U} \neq \varnothing$ for some $k \geq 1$, it follows from uniqueness that $\mu_t = \hat{\mu}_t$.

The $t = 1$ case

- For $t = 1$, we get at least one equilibrium measure, μ_1, which is an SRB measure. By the Pesin entropy formula,

$$P_g(\varphi_1) = h_{\mu_1}(g) - \int_M \log \left| dg|_{E^u(x)} \right| \, d\mu_1(x) = 0.$$

- If ν is any other equilibrium measure for φ_1, it also satisfies the entropy formula.

The $t = 1$ case

Theorem (Ledrappier, Young '84)

Any measure with positive Lyapunov exponents satisfying Pesin entropy formula is an SRB measure.

Theorem (Rodriguez-Hertz, Rodriguez-Hertz, Tahzibi, Ures '10)

A transitive surface map has at most one SRB measure.

- So if ν has positive Lyapunov exponents, ν is an SRB measure. By uniqueness of SRB measures, $\nu = \mu_1$.
- But if ν has no positive Lyapunov exponents, then $\log \left| dg|_{E^u(x)} \right| = 0$ ν-a.e., so ν is supported on the (finite) set of singularities.

The $t > 1$ case

- If $t > 1$ and ν an equilibrium state for φ_t, then by the Margulis-Ruelle inequality,

$$h_\nu(g) \le t \int \log\det \left| dg|_{E^u(x)} \right| \, d\nu(x)$$

w/ equality $\iff$ $\int \log\det \left| dg|_{E^u(x)} \right| \, d\nu(x) = 0$.

- Only measures with zero Lyapunov exponents are supported on singularities. If δ is such a measure,

$$P(\varphi_t) = h_\nu(g) + \int \varphi_t \, d\nu \le 0 = h_\delta(g) + \int \varphi_t \, d\delta$$

so we have equality, so $\log\det \left| dg|_{E^u(x)} \right| = 0$ ν-a.e., so ν is supported on the singularities.

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