SRB measures of singular hyperbolic attractors

Penn State University Dynamical Systems Seminar

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Singular hyperbolic attractors

Setting:

- M Riemannian manifold, $K \subset M$ open and precompact, $N \subset K$ closed, $N^+ = N \cup \partial K$;
- $f : K \setminus N \to K$ diffeomorphism onto its image;
- $N^- =$ image of continuous extensions of f to $N^+ \subset \overline{K}$; or more formally,

$$N^- = \left\{ y \in K : \exists z \in N^+, z_n \in K \setminus N \text{ s.t. } z_n \to z, \ f(z_n) \to y \right\}$$

- $K^+ = \{x \in K : f^n(x) \notin N^+ \ \forall n \geq 0\}$;
- $D = \bigcap_{n \geq 0} f^n(K^+)$, $\Lambda = \overline{D}$ (Λ is the *attractor* for f).
- Λ is a *singular hyperbolic attractor* if there is a continuous splitting $z \mapsto E^s(z) \oplus E^u(z)$ over $K \setminus N$ into *stable* and *unstable* subspaces. In particular, there are $C > 0$ and $\lambda > 1$ so that for any $z \in D$, $n \geq 0$:

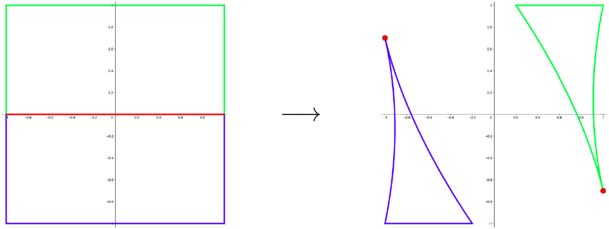
$$\begin{aligned} \|df_z^n v\| &\leq C\lambda^{-n} \|v\| \quad \forall v \in E^s(z); \\ \left\|df_z^{-n} v\right\| &\leq C\lambda^{-n} \|v\| \quad \forall v \in E^u(z). \end{aligned}$$

Example 1: Geometric Lorenz attractor

- $I = (-1, 1)$, $K = I \times I$, $N = I \times \{0\}$, $f : K \setminus N \to K$: given by $f(x, y) = \big(\varphi(x, y), \psi(x, y)\big)$, where

$$\varphi(x, y) = (\operatorname{sgn}(y) Bx|y|^{\nu} - B|y|^{\nu_0} + 1) \operatorname{sgn}(y)$$
$$\psi(x, y) = ((1 + A)|y|^{\nu_0} - A) \operatorname{sgn}(y)$$

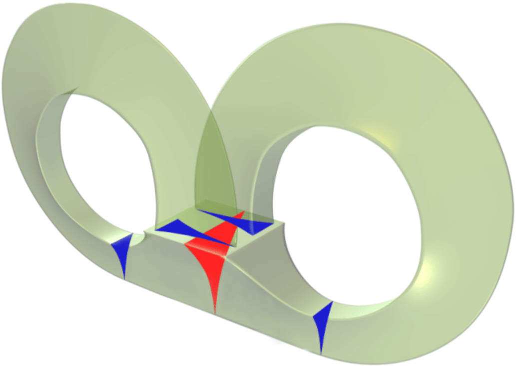
where $0 < A < 1$, $0 < B < \frac{1}{2}$, $\nu > 1$, and $1/(1 + A) < \nu_0 < 1$.



- The two dots form the set N^-, the "image" of the singular set N.

Example: Geometric Lorenz attractor

The geometric Lorenz attractor is a Poincaré first-return map for a transverse cross-section of the flow of the classical Lorenz attractor.



Example: Lorenz-type maps

More generally, a *Lorenz-type map* is a map $f : K \setminus N \to K$, $K = I \times I$, $N = I \times \{a_0, a_1, \ldots, a_{q+1}\}$, where

$$-1 = a_0 < a_1 < \cdots < a_q < a_{q+1} = 1$$

and, in addition to certain regularity conditions on df,

- $\lim_{y \uparrow a_i} f(x, y) = f_i^-$, $\lim_{y \downarrow a_i} f(x, y) = f_i^+$ ($f_i^\pm \in \overline{K \setminus N}$ constant points, independent of $x \in I$);
- $f|_{I \times (a_i, a_{i+1})} : I \times (a_i, a_{i+1}) \to K$ is a diffeomorphism onto its image;
- $\|\varphi_x\|, \left\|\psi_y^{-1}\right\| < 1$, where $f(x, y) = \big(\varphi(x, y), \psi(x, y)\big)$.

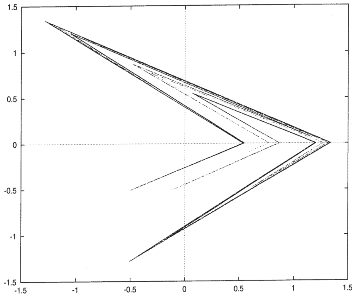
Theorem (Afraimovich, Bykov, Shilnikov '83)

If M is a compact Riemannian manifold w/ dim $M \geq 3$, *there exists a vector field X and a smooth submanifold S such that the first-return time map f induced on S by the flow given by X is a Lorenz-type map.*

Example: Lozi attractor

- Lozi map: Two-dimensional perturbation of the tent map, $f : K \setminus N \to K$, $K = (-c, c)^2$, $c \in (0, 1.5)$, $N = \{0\} \times (-c, c)$, $a > 0$, $b > 0$ sufficiently small:

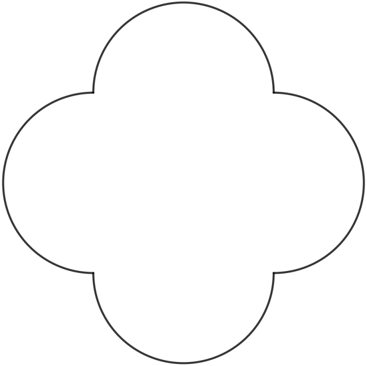
$$f(x, y) = (1 + by - a|x|, x)$$



- Topologically conjugate to the Hénon map
- In this case, the map is continuous on N, but not differentiable.

Billiards

- Say $\Omega \subset \mathbb{R}^2$ is a precompact region, $\partial\Omega$ a finite union of smooth curves. Suppose $\partial\Omega$ fails to be differentiable on a finite set Γ.
- The corresponding billiard system $T : \partial\Omega \times [0, \pi] \to \partial\Omega \times [0, \pi]$ is singular hyperbolic, with singular set $N = \Gamma \times [0, \pi]$.



SRB measures

- Suppose $f : U \to M$ is a hyperbolic map on a Riemannian manifold M. An *SRB measure* is an invariant Borel probability measure μ for which:
  - f has positive Lyapunov exponents μ-a.e., and
  - μ admits absolutely continuous conditional measures on the unstable leaves $W^u(x)$ (w.r.t. Riemannian leaf volume)
- SRB measures are hyperbolic *physical measures*: $m(\mathcal{B}_\mu) > 0$, where m is the Lebesgue/Riemannian volume and $\mathcal{B}_\mu$ is the *basin* of μ:

$$\mathcal{B}_\mu := \left\{ x : \frac{1}{n} \sum_{k=0}^{n-1} \left(\varphi \circ f^k \right) (x) \xrightarrow{n \to \infty} \int_U \varphi \, d\mu \quad \forall \varphi \in C^0 \right\}$$

Stable and unstable manifolds

Theorem (Pesin '92)

- *For m-a.e. every $z \in K \setminus N$, there are embedded submanifolds $W^s_{\text{loc}}(z)$ and $W^u_{\text{loc}}(z)$ containing z for which $T_z W^s_{\text{loc}}(z) = E^s(z)$ and $T_z W^u_{\text{loc}}(z) = E^u(z)$.*
- *Furthermore, there is an $\alpha < 1$ and $C > 0$ for which, for all $n \geq 0$, letting ρ denote Riemannian distance,*

$$\rho(f^n(x), f^n(y)) \leq C\alpha^n \rho(x, y) \quad \text{for } x, y \in W^s_{\text{loc}}(z),$$
$$\rho(f^{-n}, f^{-n}(y)) \leq C\alpha^n \rho(x, y) \quad \text{for } x, y \in W^u_{\text{loc}}(z).$$

- *The submanifolds admit hyperbolic product structure: for $w \in N \setminus K$ sufficiently close to z, the intersection $W^s_{\text{loc}}(z) \cap W^u_{\text{loc}}(w)$ is nonempty and contains exactly one point.*

Setting of main result

Setting:

- M Riemannian manifold, $K \subset M$ open and precompact, $N \subset K$ closed;
- $f : K \setminus N \to K$ diffeomorphism onto its image;
- $N^- =$ image of continuous extensions of f to $N^+ \subset \overline{K}$; or more formally,

$$N^- = \{y \in K : \exists z \in N^+, z_n \in K \setminus N \text{ s.t. } z_n \to z,\ f(z_n) \to y\}$$

  (for example, $N^- = \{f_i^\pm : 1 \leq i \leq q\}$ for Lorenz-type maps, where $f_i^+ = \lim_{y \downarrow a_i} f(x,y)$ and $f_i^- = \lim_{y \uparrow a_i} f(x,y)$);
- Λ a singular hyperbolic attractor, expansive constant $\lambda > 1$.

Main result

Assumptions:

1. N is a disjoint union of finitely many closed submanifolds $N_1, \ldots, N_m$ of dimension equal to $\mathrm{codim}(W^u)$;
2. The local unstable manifolds $W^u_{\mathrm{loc}}(x)$ intersect the singular set N transversally, with angle uniformly bounded away from 0;
3. $f^j(N^-) \cap N = \varnothing$ for $0 \leq j < k$, $\lambda^k > 2$;
4. The stable foliation is *locally continuous*: the local stable curves $W^s_{\mathrm{loc}}(z)$ (and in particular their length) vary continuously with z.

Theorem (V. 2022)

- *If $f : K \setminus N \to K$ as above satisfies Assumptions 1, 2, 3, then the attractor Λ admits finitely many ergodic SRB measures $\mu_1, \ldots, \mu_n$.*
- *If f also satisfies Assumption 4, then there is a finite collection of f-invariant subsets $\Lambda_1, \ldots, \Lambda_n$, clopen in Λ, each supporting a unique SRB measure. (In particular, if $f|_\Lambda : \Lambda \to \Lambda$ is transitive, f admits a unique SRB measure.)*

A simple non-example

- Assumption that N is a finite union of submanifolds is required for arguments. Result may not hold if N has infinitely many components.
- Example: Take a countable number of horizontal lines in $(-1, 1)^2$.
- In each resulting section, embed a copy of the geometric Lorenz attractor.
- Each section admits its own SRB measure.
- The proof of the main result requires a lower bound on the distance between components of N, which we don't have in this example.

Earlier similar result in discrete setting

Theorem (Sataev '92)

Suppose $f : K \setminus N \to K$ is a singular hyperbolic map for which we have:

1. $\exists\ B, \beta, \varepsilon_0 > 0$ *for which* $\forall n \geq 0$, $0 < \varepsilon < \varepsilon_0$:

$$\mathrm{Vol}\left(f^{-n}(B_\varepsilon(N))\right) < B\varepsilon^\beta$$

   where $B_\varepsilon(N) := \{z \in K : \rho(z, N) < \varepsilon\}$*;*

2. $\exists\ \beta, \varepsilon_1 > 0$ *for which: whenever* $V \subset K$ *is a manifold tangent to the unstable cone,* $\exists\ B_0 = B_0(V)$ *s.t.* $\forall 0 < \varepsilon < \varepsilon_1$*:* $\forall n \geq 0$*:*

$$\mathrm{Vol}_V\left(V \cap f^{-n}(B_\varepsilon(N))\right) \leq B_0\varepsilon^\beta.$$

Then there are finitely many disjoint invariant sets $\Lambda_1, \ldots, \Lambda_n$ *and ergodic SRB measures* $\mu_1, \ldots, \mu_n$ *with* $\mu_i(\Lambda_i) = 1$ *for each* i*, and* $f|_{\Lambda_i}$ *topologically transitive.*

Earlier similar result in discrete setting

- Recall our assumptions:
  1. N is a disjoint union of finitely many closed submanifolds $N_1, \ldots, N_m$ of dimension equal to $\mathrm{codim}(W^u)$;
  2. The local unstable manifolds $W^u_{\mathrm{loc}}(x)$ intersect the singular set N transversally, with angle uniformly bounded away from 0;
  3. $f^j(N^-) \cap N = \varnothing$ for $0 \leq j < k$, $\lambda^k > 2$;

Theorem (Pesin '92)

In dimension 2, these assumptions imply the recurrence hypotheses of previous theorem.

- These three assumptions do not give existence in dimensions higher than 2. However, if hold for a higher-dimensional singular hyperbolic map, these assumptions still imply that there are *at most* finitely many SRBs, and the proof avoids using recurrence assumptions.

Singular hyperbolic flows

- Let M be a 3-dimensional Riemannian manifold, X a C^r vector field with flow X_t.
- A compact X_t-invariant subset $\Lambda \subset M$ is *singular hyperbolic* if
  1. Λ is *partially hyperbolic*, in that Λ admits an invariant splitting $T_\Lambda M = E^s \oplus E^c$ for which $d(X_t)|_{E^s}$ is contracting, E^s dominates E^c, and E^c is volume-expanding: there are $K > 0$ and $\lambda > 0$ for which:
     $$\begin{aligned} \|d(X_t)|_{E^s}\| &\leq Ke^{-\lambda t}; \\ \|d(X_t)|_{E^s}\| \cdot \|d(X_t)|_{E^c}\| &\leq Ke^{-\lambda t}; \\ |J(d(X_t)|_{E^c})| &\geq Ke^{\lambda t} \quad \text{(where } J \text{ is the Jacobian)} \end{aligned}$$
  2. all singularities of X contained in Λ are hyperbolic (here "singularities" just means fixed points of the flow, or points $z \in M$ where $X(z) = 0$).
- The standard Lorenz system is an example of such a flow.

Spectral decomposition for flows

Theorem (Pacifico, Morales '07)

A singular hyperbolic attractor for a flow with dense periodic orbits and a unique singularity is a finite union of transitive sets. (If the vector field is "Kupka-Smale", this union is disjoint.)

- "*Kupka-Smale*" essentially means that the closed orbits and critical points are hyperbolic, and the stable manifold of a critical point can intersect the unstable manifold of another critical point only transversally.

Theorem (Araújo, Pacifico, Pujals, Viana '07)

A transitive attracting set Λ_i as above supports a unique SRB measure (physical probability measure which disintegrates to an absolutely continuous measure along center-unstable leaves).

Spectral decomposition for flows

- A consequence of these results is that a singular hyperbolic attractor Λ for a flow X on a 3-manifold M with dense periodic points is the following spectral decomposition:
  - $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$, with Λ_i compact, X_t-invariant, and transitive;
  - There are exactly k ergodic SRB measures $\mu_1, \ldots, \mu_k$, with μ_i supported on Λ_i. Every SRB is a convex combination of the μ_i.

Theorem (Sataev '10)

If $\Lambda \subset M$ is a singular hyperbolic attractor for a flow X on a Riemannian n-manifold M whose stable distribution E^s has dimension $n-2$, then Λ admits a spectral decomposition as above.

Theorem (Araújo '22)

Let X be a vector field on a 3-manifold, Λ a singular hyperbolic attractor, s_L the number of hyperbolic singularities of X contained in Λ. Then the number s of ergodic SRB measures on Λ whose support contains a singularity is $s \leq 2s_L$.

Uniform hyperbolicity setting

- Let $f : M \to M$ be a diffeomorphism of a Riemannian manifold M. An invariant compact subset $\Lambda \subset M$ is **uniformly hyperbolic** if there is a continuous invariant splitting $T_\Lambda M = E^s \oplus E^u$ for which there are $C > 0$ and $\lambda > 1$ so that for any $z \in \Lambda$, for all $n > 0$, we have

$$|df_z^n v| \geq C\lambda^n |v| \quad \forall z \in \Lambda v \in C^u(z);$$
$$|df_z^{-n}| \geq C\lambda^n |v| \quad \forall z \in \Lambda, v \in C^s(z).$$

- This is similar to singular hyperbolic case, but where this splitting occurs *everywhere* in the domain. (Note this splitting holds in the set D by definition, but does not extend to $\overline{D} = \Lambda$ since $\Lambda \cap N \neq \varnothing$.)
- Besides the discontinuity/non-differentiability, the differential may be *unbounded* near the singular line. This obstructs classical arguments for existence/uniqueness of SRB measures.

Uniform hyperbolicity spectral decomposition

Theorem (Smale '67, Bowen '71, Sinai '72, Ruelle '73, Ruelle '76)

Let $f : M \to M$ be an Axiom A diffeomorphism attractor Λ. Then Λ admits a decomposition $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_s$ into finitely many pairwise disjoint closed invariant sets for which

1. $f|_{\Lambda_i} : \Lambda_i \to \Lambda_i$ *is topologically transitive* $\forall i$;
2. *each* Λ_i *admits a unique ergodic SRB measure.*

- Key ingredient in Bowen's proof is construction of a Markov partition.
- Markov partition relies on *local hyperbolic product structure*.

Theorem (Bowen '71)

For any $\varepsilon > 0$, there is a $\delta > 0$ for which for any $x, y \in \Lambda$ with $\rho(x, y) < \delta$, with Λ a uniformly hyperbolic attractor, $W^s_\varepsilon(x) \cap W^u_\varepsilon(y)$ is nonempty and consists of a single point $[x, y]$.

Hyperbolic product structure

Theorem (Pesin '92)

There is an $\varepsilon > 0$ such that for Leb-a.e. $z \in K \setminus N$, there are $\delta_1 > \delta_2 > \delta_3 > 0$ for which for $y \in K \setminus N$ with $\rho(y,z) < \delta_3$, $W^u_\varepsilon(y) \cap W^s_{\delta_1}(z)$ contains a single point, denoted $[y,z]$; and furthermore, letting $B^u([y,z],\delta_2)$ denote the ball of radius δ_2 in $W^u_{\mathrm{loc}}([y,z])$, we have $B^u([y,z],\delta_2) \subset W^u_\varepsilon(y)$.

- In the uniformly hyperbolic setting, δ_1 and δ_3 *are independent of the base point z*. In particular, they are bounded below. This ensures that we have a *finite* Markov partition.
- The singular set disrupts the length of the local stable and unstable manifolds. We don't get uniform length of stable leaves.

Construction of SRB measures

- Define $W^u(z) = \bigcup_{n\geq 0} f^n\left(W^u_{\text{loc}}(f^{-n}(z))\right)$ for $z \in K$ ($W^s(z)$ is defined analogously for $z \in \Lambda$).
- Let $J^u(z) = \det\left(df|_{E^u(z)}\right)$ denote the unstable Jacobian of f at a point $z \in \Lambda$. For $y \in W^u(z)$, set

$$\kappa(z, y) = \prod_{j=0}^{\infty} \frac{J^u\left(f^{-j}(z)\right)}{J^u\left(f^{-j}(y)\right)}$$

- Let m^u_z and ρ^u_z be the Riemannian leaf volume and leaf metric on $W^u(z)$. Let $U_0 := B^u(z, r) \subset W^u_{\text{loc}}(z)$ be the disc of ρ^u_z-radius r centered at z.
- Finally let $U_n = f(U_{n-1}) \setminus N^+$.

Construction of SRB measures (cont.)

- Define the measures $\widetilde{\nu}_n$ on $U_n \subset W^u(f^n(z))$ by

$$d\widetilde{\nu}_n(y) = \widetilde{C}_n(z)\kappa(f^n(z), y)dm_z^u(y),$$

  where $\widetilde{C}_n(z)$ is a normalizing factor.
- Let ν_n be the measure on Λ given by $\nu_n(A) = \widetilde{\nu}_n(A \cap U_n)$ for Borel $A \subset \Lambda$.
- Each ν_n is defined only on subsets of a particular unstable manifold $W^u(f^n(z))$.
- Define:

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \nu_k$$

- The final step in the construction is to show μ_n has an f-invariant weak limit measure μ concentrated on D. (Note μ may depend on the reference point z.)

Local continuity of stable foliation

- Is there a connection between the number/supports of SRB measures and the topological properties of the dynamics (e.g. topological transitivity/transitive components)?
- Given $z \in K \setminus N$, there is a radius $r = r(z) > 0$ for which $W^s_{\mathrm{loc}}(z) \cap B_r(z)$ is the graph of a C^1 function $\psi_z : U_z \to M$, $U_z \subset T_z W^s_{\mathrm{loc}}(z)$.
- W^s is *locally continuous* if $y \mapsto \psi_y$ and $(y, u) \mapsto d(\psi_y)_u$ are continuous over $y \in (K \setminus N) \cap B_{r(x)}(x)$, $u \in U_y \subset T_y W^s(y)$. (Note ψ_y is defined μ-a.e. in $(K \setminus N) \cap B_{r(x)}(x)$, μ any SRB measure.)
- In particular, the maximal radius $z \mapsto r(z)$ in which $W^s_{\mathrm{loc}}(z) \cap B_{r(z)}(z)$ is a C^1 curve varies continuously over Λ.

Transitivity and ergodicity

Theorem (Pesin '92)

Let $f : K \setminus N \to K$ be singular hyperbolic.

- *There is a countable collection of f-invariant subsets $\{U_i\}_{i\geq 1}$, open in Λ, for which $\overline{\bigcup_{i\geq 1} U_i} = \Lambda$, and each of which is supported by exactly one ergodic SRB measure.*
- *If $f|_\Lambda : \Lambda \to \Lambda$ is topologically transitive, then $U_1 = \Lambda$ is the only member of this collection, and thus the ergodic SRB measure is unique.*

- Under earlier assumptions (finitely many of components of N, transverse intersection of W^u with N, sufficiently long recurrence to N), $f : K \setminus N \to K$ admits at most finitely many ergodic SRB measures.
- It follows that under the additional assumption of local continuity of W^s, then this collection of sets $\{U_i\}_{i\geq 1}$ is finite.

Construction of ergodic components

- Let μ be an SRB measure for Λ. For μ-a.e. $z \in \Lambda$, $W^u_{\text{loc}}(z) \subset \Lambda$ and the stable discs $B^s_{r(y)}(y)$ are defined for m^u_z-a.e. $y \in W^u_{\text{loc}}(z)$, i.e. on a set $A^u(z) \subset W^u_{\text{loc}}(z)$ of full m^u_z-measure. (Recall m^u_z is the Riemannian leaf volume of $W^u_{\text{loc}}(z)$.)
- Define the set

$$Q(z) = \left(\bigcup_{y \in A^u(z)} B^s_{r(y)}(y) \right) \cap \Lambda \cap B_{r(z)}(z).$$

Note $\mu(Q(z)) > 0$. (Recall $r(z)$ is the maximal radius in which $W^s_{\text{loc}}(z) \cap B_{r(z)}(z)$ is a C^1 curve.)
- Note $Q = \bigcup_{n \in \mathbb{Z}} f^n(Q(z))$ is f-invariant, and thus an ergodic component of μ.
- Openness of Q (mod 0) in Λ follows from the local continuity of W^s (i.e. continuity of $y \mapsto r(y)$ for $y \in W^u_{\text{loc}}(z)$).

Proof of finiteness: Preliminary constructions

- Recall $K^+ = \{x \in K : f^n(x) \notin N^+ \; \forall n \geq 0\}$ and $D = \bigcap_{n \geq 0} f^n(K^+)$.
- Given $\delta > 0$, let $B_\delta^- \subset D$ consist of those $x \in D$ for which $W_\delta^u(y)$ exists and contains x, for some $y \in D$.
- Suppose $\delta_1 < \delta_2$. Then $B_{\delta_2}^- \subseteq B_{\delta_1}^-$.
  - Indeed, if $x \in B_{\delta_2}^-$, then $x \in W_{\delta_2}^u(y)$ for some $y \in D$.
  - By certain regularity hypotheses, $D \cap W_{\delta_2}^u(y)$ has full measure, so can pick $y' \in W_{\delta_2}^u(y)$ that is with δ_1-distance to x.
  - Follows that $x \in B_{\delta_1}^-$.

Lemma

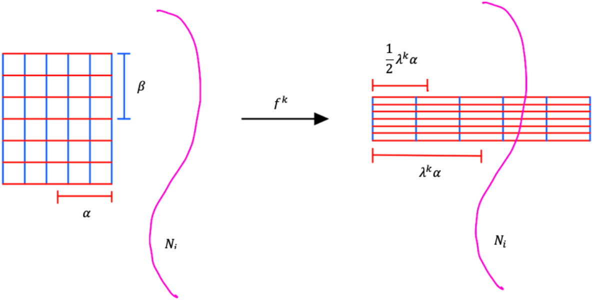
There exists a $\delta_0 > 0$ so that if μ is an ergodic SRB measure of $f : \Lambda \to \Lambda$, then $\mu(B_{\delta_0}^-) > 0$.

Proving first lemma

- B_δ^- is the set of points whose local unstable leaves have radius $\delta > 0$.
- Recall $N = \bigcup_{i=1}^m N_i$, and if U is a neighborhood of N, then $f(U)$ is a neighborhood of N^-.
- Since $f^j(N^-) \cap N = \varnothing$ for $1 \leq j < k$, $\lambda^k > 2$ ($\lambda > 1$ expansive constant), and N and $f^k(N^-)$ are closed, there is a radius $Q > 0$ so that:
    - the open neighborhoods $B_Q(N_i)$, $1 \leq i \leq m$, are disjoint;
    - $f^j(B_Q(N_i)) \cap N = \varnothing$ for $1 \leq j < k$.
- We let $\delta_0 < Q$.
- Choose ergodic SRB measure μ, and μ-generic point $x \in D$. Using hyperbolic product structure, construct rectangle R of stable leaves of radius $\beta > 0$ and unstable leaves of radius $\alpha > 0$. Then $\mu(R) > 0$.
- $f(R)$ is a rectangle whose unstable leaves have length $\lambda\alpha$.

Proving first lemma (cont)

- **Idea:** Use iterates of R to extend the unstable leaves until they are of radius $\geq \delta_0$. Once this happens, $f^j(R) \subset B^-_{\delta_0}$, and $\mu(B^-_{\delta_0}) > \mu(f^j(R)) = \mu(R) > 0$.
- **Obstruction:** What if $f^j(R) \cap N \neq \varnothing$ before $\lambda^j \alpha > \delta_0$?
- Since $\delta_0 > Q$, $Q =$ radius of disjoint neighborhoods of components of N, $f^j(R)$ lies in one of these neighborhoods.
- Choose new rectangle R_1 of radius $\alpha_1 \geq \frac{1}{2}\lambda^j \alpha$.



Proving first lemma (cont)

- Iterate R_1 until either:
  - $\lambda^j \alpha_1 \geq \delta_0$ (in which case $\mu(B^-_{\delta_0}) > 0$, and we're done); or
  - $f^{j_1}(R_1) \cap N \neq \varnothing$ for some $j_1 \geq k$ (since $R_1 \subset B_Q(N_i)$).
- In latter case, take new rectangle $R_2 \subset f^{j_1}(R_1) \setminus N$ with unstable leaves of radius $\alpha_2 \geq \frac{1}{2}\lambda^{j_1}\alpha_1$.
- Repeat this process. Each time a rectangle intersects N, we take a leaf of at least half the radius, creating a sequence of rectangles $\{R_\ell\}$ with unstable leaves of radii

$$\alpha_\ell \geq \frac{1}{2^\ell}\lambda^{j_1+\cdots+j_\ell}\alpha_1 > \frac{\lambda^{k\ell}}{2^\ell}\alpha_1 = \left(\frac{\lambda^k}{2}\right)^l \alpha_1,$$

  with each $j_\ell \geq k$ the time it takes for R_ℓ to intersect N.
- Since $\lambda^k > 2$, this will eventually exceed δ_0, at which point $0 < \mu(R_\ell) \leq \mu(B^-_{\delta_0})$. □

Proving finiteness

- The main result is proven once we show $B^-_{\delta_0}$ is charged by at most finitely many ergodic SRB measures.
- Let $\Lambda^\pm \subset \Lambda$ be the points on which the limits

$$\varphi_\pm(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi|_\Lambda \left(f^{\pm k}(x) \right)$$

both exist for every $\varphi \in C^0(K)$. Then $\mu(\Lambda^\pm) = 1$ by Birkhoff ergodic theorem, w.r.t. any invariant μ.
- Recall $x \in B^-_\delta \iff \exists y = y(x) \in D^-_\ell$ s.t. $x \in W^u_\delta(y)$. So let Λ^0 be the set of those points $x \in B^-_\delta$ for which, letting $y = y(x)$:

$$\exists A^u(y) \subset W^u_\delta(y) \text{ of full } \nu^u\text{-measure s.t. } A^u(y) \subset \Lambda^+$$

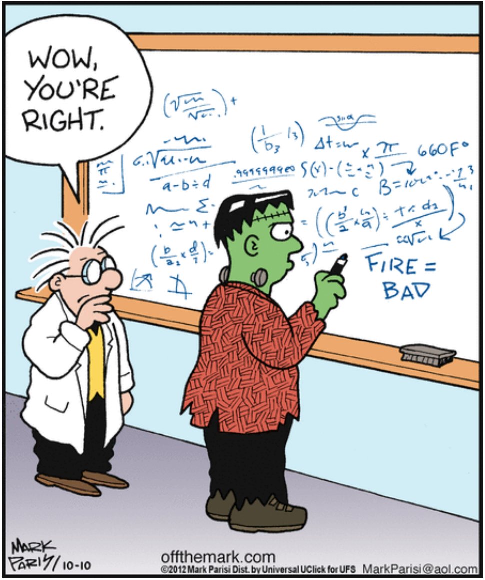
and

$$\varphi_+|_{A^u(y)} \text{ is constant } \forall \varphi \in C^0(K).$$

Proving finiteness: Hopf argument

- We make the following two claims:
  1. Λ^0 has full measure in B_δ^- for any f-invariant measure, and
  2. Λ^0 is closed.
- Granting these claims, partition Λ^0 into equivalence classes on which φ_+ is constant. More precisely, $x \sim x'$ for $x, x' \in \Lambda^0$ if $\varphi_+(A^u(y)) = \varphi_+(A^u(y'))$ for all $\varphi : K \to \mathbb{R}$ continuous ($y = y(x)$, $y' = y(x')$ as above).
- Idea: Use hyperbolic product structure and absolute continuity of holonomy map to show the equivalence classes are open in Λ^0.
- Since Λ^0 is itself closed, there may only be finitely many such equivalence classes.
- Since Λ^0 has full measure in B_δ^- for any f-invariant measure, every ergodic SRB measure must charge exactly one such equivalence class.

Thank you!


WOW, YOU'RE RIGHT.
FIRE = BAD
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