SRB measures and topological singular hyperbolic attractors

International and Online Mathematics Days III

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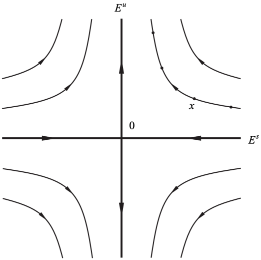
December 9 2022

Hyperbolic sets

- Let M be a Riemannian manifold, $U \subset M$ an open set, $f : U \to M$ a diffeomorphism onto its image.
- A compact f-invariant $\Lambda \subset U$ is a *hyperbolic set* if there exist constants $\lambda > 1$ and $c > 0$ such that each $x \in \Lambda$ admits a splitting $T_x M = E^s(x) \oplus E^u(x)$ for which
  - $df_x(E^s(x)) = E^s(f(x))$ and $df_x(E^u(x)) = E^u(f(x))$;
  - $\|df_x^n(v)\| \leq c\lambda^{-n}\|v\| \ \forall v \in E^s(x)$;
  - $\|df_x^{-n}(f)\| \leq c\lambda^{-n}\|v\| \ \forall v \in E^u(x)$.
- In this setting, we say f is *uniformly hyperbolic* on Λ.

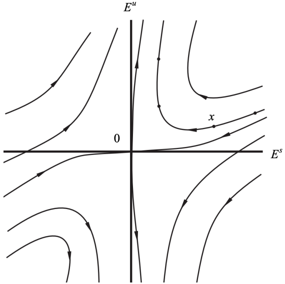
Example: A hyperbolic fixed point of a linear map (or flow) $\mathbb{R}^2 \to \mathbb{R}^2$:



Stable and unstable submanifolds

- For each $x \in \Lambda$, there are embedded discs W^s_{loc} and W^u_{loc} of U containing x so that $T_x W^s_{\text{loc}}(x) = E^s(x)$ and $T_x W^u_{\text{loc}}(x) = E^u(x)$, and for which there is a $C > 0$, $0 < \alpha < 1$ such that for $x \in U$,
  - $f(W^s_{\text{loc}}(x)) = W^s_{\text{loc}}(f(x))$ and $f(W^u_{\text{loc}}(x)) = W^u_{\text{loc}}(f(x))$;
  - $\rho(f^n(x), f^n(y)) \leq C\alpha^n \rho(x, y)\ \forall y \in W^s_{\text{loc}}(x)$;
  - $\rho(f^{-n}(x), f^{-n}(y)) \leq C\alpha^n \rho(x, y)\ \forall y \in W^u_{\text{loc}}(x)$.
- We call $E^s(x)$, $E^u(x)$ the *stable/unstable subspaces* and $W^s_{\text{loc}}(x)$, $W^u_{\text{loc}}(x)$ the *stable/unstable submanifolds* at $x \in U$.

Example: A hyperbolic fixed point of a nonlinear map (or flow) $\mathbb{R}^2 \to \mathbb{R}^2$:

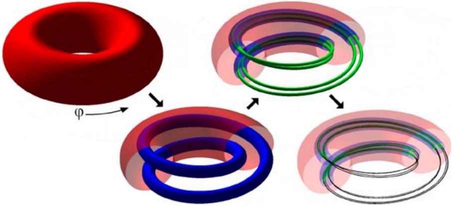


Example: Smale-Williams Solenoid

- Consider a map $F : \mathbb{S}^1 \times \mathbb{D} \to \mathbb{S}^1 \times \mathbb{D}$ of the solid torus given by

$$F(\varphi, x, y) = \left(2\varphi,\ \frac{1}{2}\cos(\varphi) + \frac{1}{5}x,\ \frac{1}{2}\sin(\varphi) + \frac{1}{5}y\right)$$

- F is expanding in the $\mathbb{S}^1$ direction, and contracting in the $\mathbb{D}$ direction.



- The *attractor* is the set $\Lambda = \bigcap_{n\geq 0} F\left(\mathbb{S}^1 \times \mathbb{D}\right)$. Topologically, it carries all of the asymptotic information about F.
- More generally, an *attractor* of a topological dynamical system $f : \Omega \to \Omega$ is a subset $\Lambda \subset \Omega$ for which there is an open $U \supset \Lambda$ for which $\bigcap_{n\geq 0} f^n(U) = \Lambda$. (In general, it may not be unique!)

Ergodicity and Transitivity

- How do we observe "long-term behavior"?
- A *topological attractor* of a dynamical system $f : U \to M$ is a closed subset of the form $\Lambda = \bigcap_{n \geq 0} f^n(U)$, i.e., "where points end up."
- But does the attractor resemble *every* orbit? Does every orbit "look like" the attractor?
- If $K \subset M$ is f-invariant (i.e., $f(K) \subset K$), f is *transitive on* K if for every pair of open subsets $V_1, V_2 \subset K$, there is an $n \geq 1$ for which $f^n(V_1) \cap V_2 \neq \varnothing$.
  - Transitive dynamical systems are *topologically indecomposable.*
- Suppose μ is an f-invariant probability measure on M (i.e., $\mu(f^{-1}(A)) = \mu(A)$ for all $A \subset M$). f is *ergodic* if every invariant subset $K \subset M$ has $\mu(K) = 1$ or $\mu(K) = 0$.
  - Ergodic dynamical systems are *measurably indecomposable.*

Ergodic theory and probability theory

- If a dynamical system $f : U \to M$ has sensitive dependence on initial conditions, computing orbits explicitly and for all time is often impossible. But the *statistical behavior* is often quite regular.
- Recall that a sequence of identically distributed random variables $X_0, X_1, X_2, \ldots$ with expectation $\mathbb{E}[X_1] < \infty$ satisfies the *law of large numbers* if
$$\mathbb{P}\left[\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{n\to\infty} \mathbb{E}[X_1]\right] = 1.$$
- If $\varphi : M \to \mathbb{R}$ and μ is an f-invariant probability measure on M, then $(\varphi \circ f^n)_{n\geq 0}$ is a sequence of identically distributed random variables.

Ergodic theory and probability theory

Theorem

f Ω is a compact metrizable space, $f : \Omega \to \Omega$ measurable, μ an f-inv. prob. measure on Ω with respect to which f is ergodic, then:

$$\mu\left\{x \in \Omega : \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)) \xrightarrow{n\to\infty} \int_\Omega \varphi \, d\mu \, \forall \varphi \in C^0(\Omega)\right\} = 1.$$

- In other words, *ergodic dynamical systems obey the law of large numbers for every $\varphi : \Omega \to \mathbb{R}$:*

$$\mathbb{P}\left[\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow{n\to\infty} \mathbb{E}[X_1]\right] = 1.$$

Physical measures

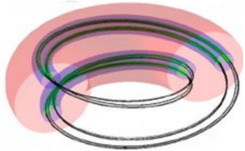
- But many invariant measures are not ergodic.
- If μ is f-invariant, its *basin* is

$$\mathcal{B}_\mu = \left\{ x \in \Omega : \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \xrightarrow{n\to\infty} \int_\Omega \varphi \, d\mu \; \forall \varphi \in C^0(\Omega) \right\}.$$

  - $\mathcal{B}_\mu$ is the set of points whose long-term behavior is described *statistically* by the measure μ.
- The *basin* of a topological attractor $\Lambda \subset \Omega$, on the other hand, is the largest open set $\mathcal{B}_\Lambda$ of points for which $\bigcap_{n\geq 0} f^n(\mathcal{B}_\Lambda) = \Lambda$.
  - $\mathcal{B}_\Lambda$ is the set of points whose long-term behavior is described *topologically* by the attractor Λ.
- Suppose our dynamical system is $f : U \to M$, M a Riemannian manifold, $U \subset M$ open, μ an f-invariant measure on M, m the Riemannian measure on M (typically not f-invariant). Then μ is a *physical measure* if $m(\mathcal{B}_\mu) > 0$ (long-term behavior is "physically observable").

SRB measures

- In the case of *hyperbolic dynamical systems* $f : U \to M$, what do physical measures "look like" (i.e. what are their supports)?
- Intuitively, they're supported on hyperbolic attractors.
  - Recall that for the Smale-Williams solenoid (and hyperbolic attractors Λ in general), $W^u_{\text{loc}}(z) \subset \Lambda$.



- An f-inv. physical measure μ is a *Sinai-Ruelle-Bowen (SRB) measure* if the conditional measures on the unstable leaves $W^u_{\text{loc}}(z)$ are absolutely continuous with respect to Lebesgue measure.
- For the solenoid, locally the SRB measure is a product of Lebesgue measure on the circle, and the Bernoulli measure on the Cantor set.
- For diffeomorphisms of compact manifolds $f : M \to M$ that are uniformly hyperbolic everywhere (*Anosov diffeomorphisms*), the SRB measure is equivalent to Lebesgue.

SRB measures for hyperbolic maps

Theorem (Sinai '72, Bowen, Ruelle '75, Ruelle '76)

Suppose M is compact, $f : U \to M$ is $C^{1+\alpha}$ and $\Lambda = \bigcap_{n \geq 0} f^n(U)$ is uniformly hyperbolic. Then there are at most finitely many ergodic SRB measures on Λ. If, furthermore, $f|_\Lambda$ is topologically transitive, then there is a unique SRB measure μ on Λ, and $\mathcal{B}_\mu$ has full measure in U.

Theorem (Hu, Young '95)

There is a C^2 diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ that is nonuniformly hyperbolic (i.e., the rate of expansion in the direction E^u is nonuniform) and does not admit an SRB measure.

Theorem (Rodriguez-Hertz, Rodriguez-Hertz, Tahzibi, Ures '10)

If $f : M \to M$ is a topologically transitive $C^{1+\alpha}$ diffeomorphism of a surface M, then f admits at most one SRB measure.

Singular hyperbolic attractors

Setting:

- M Riemannian manifold, $K \subset M$ open and precompact, $N \subset K$ closed, $N^+ = N \cup \partial K$;
- $f : K \setminus N \to K$ diffeomorphism onto its image;
- $N^- =$ image of continuous extensions of f to $N^+ \subset \overline{K}$; or more formally,

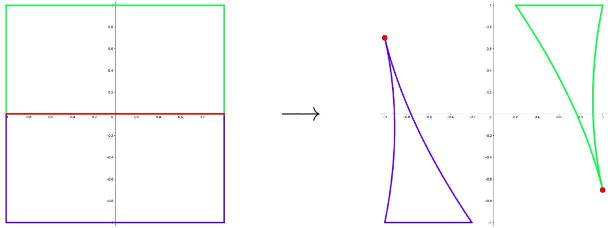
$$N^- = \left\{ y \in K : \exists z \in N^+, z_n \in K \setminus N \text{ s.t. } z_n \to z,\ f(z_n) \to y \right\}$$

- $K^+ = \{x \in K : f^n(x) \notin N^+ \ \forall n \geq 0\}$;
- $D = \bigcap_{n \geq 0} f^n(K^+)$, $\Lambda = \overline{D}$ (Λ is the *attractor* for f).
- Λ is a *singular hyperbolic attractor* if there is a continuous splitting $z \mapsto E^s(z) \oplus E^u(z)$ over $K \setminus N$ into *stable* and *unstable* subspaces. In particular, there are $C > 0$ and $\lambda > 1$ so that for any $z \in D$, $n \geq 0$:

$$\begin{aligned} \left\| df_z^n v \right\| &\leq C\lambda^{-n} \left\| v \right\| \quad \forall v \in E^s(z); \\ \left\| df_z^{-n} v \right\| &\leq C\lambda^{-n} \left\| v \right\| \quad \forall v \in E^u(z). \end{aligned}$$

Example 1: Geometric Lorenz attractor

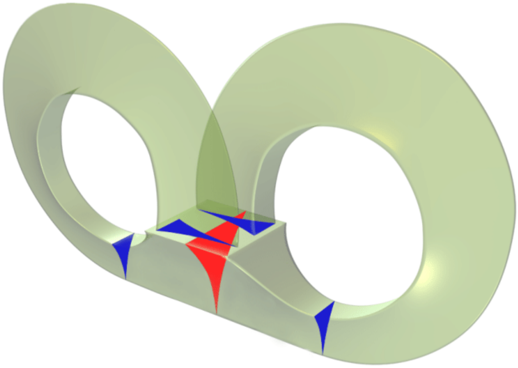
- $I = (-1,1)$, $K = I \times I$, $N = I \times \{0\}$, $f : K \setminus N \to K$ diffeomorphism onto its image given by $f(x,y) = \big(\varphi(x,y), \psi(x,y)\big)$, where:
  - $\|\varphi_x\|, \left\|\psi_y^{-1}\right\| < 1$;
  - $\lim_{y\uparrow 0} f(x,y) = f_0^-$, $\lim_{y\downarrow y} f(x,y) = f_0^+$ constant points:



- Here, the vertical lines form the unstable distribution E^u, and the horizontal lines form the stable distribution E^s.
- The two dots $f_0^\pm$ form the set N^-, the "image" of the singular set N.

Example: Geometric Lorenz attractor

The geometric Lorenz attractor is a Poincaré first-return map for a transverse cross-section of the flow of the classical Lorenz attractor.



Example: Lorenz-type maps

More generally, a *Lorenz-type map* is a map $f : K \setminus N \to K$, $K = I \times I$, $N = I \times \{a_0, a_1, \dots, a_{q+1}\}$, where

$$-1 = a_0 < a_1 < \cdots < a_q < a_{q+1} = 1$$

and, in addition to certain regularity conditions on df,

- $f : K \setminus N \to K$ is a diffeomorphism onto its image;
- $\lim\limits_{y \uparrow a_i} f(x, y) = f_i^-$, $\lim\limits_{y \downarrow a_i} f(x, y) = f_i^+$ ($f_i^\pm \in K$ constant points, independent of $x \in I$);
- $\|\varphi_x\|, \left\|\psi_y^{-1}\right\| < 1$, where $f(x, y) = \big(\varphi(x, y), \psi(x, y)\big)$.

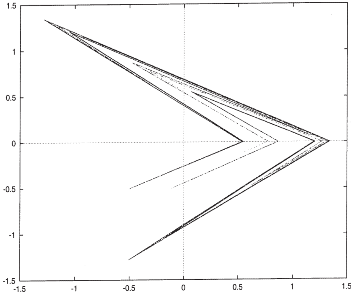
Theorem (Afraimovich, Bykov, Shilnikov '83)

If M is a compact Riemannian manifold w/ $\dim M \geq 3$, there exists a vector field X and a smooth submanifold S such that the first-return time map f induced on S by the flow given by X is a Lorenz-type map.

Example: Lozi attractor

- *Lozi map*: $f : K \setminus N \to K$, $K = (-c, c)^2$, $c \in (0, 1.5)$, $N = \{0\} \times (-c, c)$, $a > 0$, $b > 0$ sufficiently small:

$$f(x, y) = (1 + by - a|x|, x)$$



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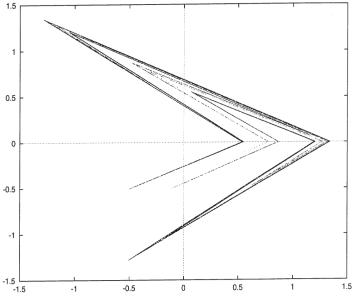

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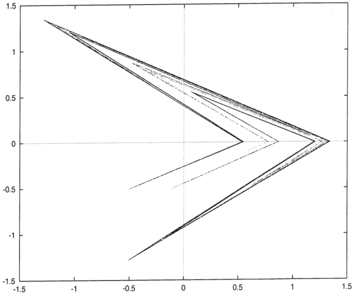

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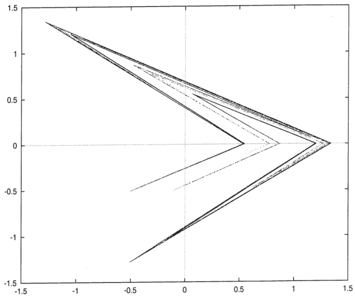


- A small perturbation of the tent map $x \mapsto 1 - a|x|$.
- f is topologically conjugate to the *Hénon map* (quadratic instead of absolute value).
- Lozi map *does not* arise as the time 1 map of a hyperbolic flow.

Setting of main result

Setting:

- M Riemannian manifold, $K \subset M$ open/precompact, $N \subset K$ closed;
- $f : K \setminus N \to K$ diffeomorphism onto its image;
- $N^- =$ image of continuous extensions of f to $N^+ \subset \overline{K}$; or more formally,

$$N^- = \left\{ y \in K : \exists z \in N^+, z_n \in K \setminus N \text{ s.t. } z_n \to z,\ f(z_n) \to y \right\}$$

  (for example, $N^- = \{f_i^\pm : 1 \le i \le q\}$ for Lorenz-type maps, where $f_i^+ = \lim\limits_{y \downarrow a_i} f(x,y)$ and $f_i^- = \lim\limits_{y \uparrow a_i} f(x,y)$);
- Λ a singular hyperbolic attractor, expansive constant $\lambda > 1$.

Theorem (Pesin '92)

Suppose $f : K \setminus N \to K$ admits a singular hyperbolic attractor Λ. Then with certain regulatory assumptions, there are at most countably many ergodic SRB measures supported on Λ.

Main result

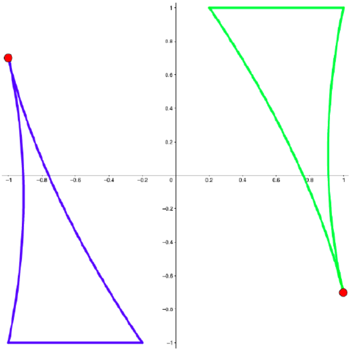
Assumptions:

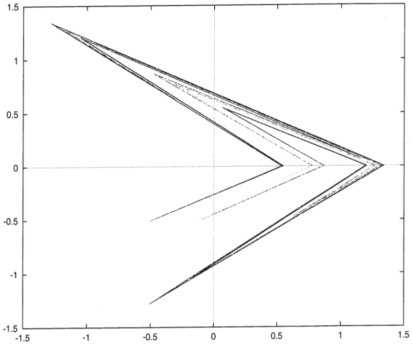
1. N is a disjoint union of finitely many closed submanifolds $N_1, \ldots, N_m$ of dimension equal to $\mathrm{codim}(W^u_{\mathrm{loc}})$;
2. The local unstable manifolds $W^u_{\mathrm{loc}}(x)$ intersect the singular set N transversally, with angle uniformly bounded away from 0;
3. $f^k(N^-) \cap N = \varnothing$ for $0 \leq k < \ell$, $\lambda^\ell > 2$;
4. The stable foliation is *locally continuous*: the local stable curves $W^s_{\mathrm{loc}}(z)$ (and in particular their length) vary continuously with z.

Theorem (V. '22)

- *If $f : K \setminus N \to K$ as above satisfies Assumptions 1, 2, 3, then the attractor Λ admits finitely many ergodic SRB measures $\mu_1, \ldots, \mu_n$.*
- *If f also satisfies Assumption 4, then there is a finite collection of f-invariant subsets $\Lambda_1, \ldots, \Lambda_n$, clopen in Λ, each supporting a unique SRB measure. (In particular, if $f|_\Lambda : \Lambda \to \Lambda$ is transitive, f admits a unique SRB measure.)*

- This implies, in particular, that the attractors for both Lorenz-type maps and the Lozi map admit finitely many SRB measures. (Recall that $W^u_{\mathrm{loc}}(x)$ lies inside the attractor Λ for all $x \in \Lambda$ where $W^u_{\mathrm{loc}}(x)$ is defined.)





A simple non-example

- Assumption that N is a finite union of submanifolds is required for arguments. Result may not hold if N has infinitely many components.
- Example: Take a countable number of horizontal lines in $(-1,1)^2$.
- In each resulting section, embed a copy of the geometric Lorenz attractor.
- Each section admits its own SRB measure.
- The proof of the main result requires a lower bound on the distance between components of N, which we don't have in this example.

Earlier similar result in discrete setting

Theorem (Sataev '92)

Suppose $f : K \setminus N \to K$ is a singular hyperbolic map for which we have:

1. $\exists\ B, \beta, \varepsilon_0 > 0$ *for which* $\forall n \geq 0$, $0 < \varepsilon < \varepsilon_0$:

$$\mathrm{Vol}\left(f^{-n}(B_\varepsilon(N))\right) < B\varepsilon^\beta$$

   where $B_\varepsilon(N) := \{z \in K : \rho(z, N) < \varepsilon\}$;

2. $\exists\ \beta, \varepsilon_1 > 0$ *for which: whenever* $V \subset K$ *is a manifold tangent to the unstable cone,* $\exists\ B_0 = B_0(V)$ *s.t.* $\forall 0 < \varepsilon < \varepsilon_1$: $\forall n \geq 0$:

$$\mathrm{Vol}_V\left(V \cap f^{-n}(B_\varepsilon(N))\right) \leq B_0\varepsilon^\beta.$$

Then there are finitely many disjoint invariant sets $\Lambda_1, \ldots, \Lambda_n$ and ergodic SRB measures $\mu_1, \ldots, \mu_n$ with $\mu_i(\Lambda_i) = 1$ for each i, and $f|_{\Lambda_i}$ topologically transitive.

Earlier similar result in discrete setting

- Recall our assumptions:
  1. N is a disjoint union of finitely many closed submanifolds $N_1, \dots, N_m$ of dimension equal to $\operatorname{codim}(W^u)$;
  2. The local unstable manifolds $W^u_{\mathrm{loc}}(x)$ intersect the singular set N transversally, with angle uniformly bounded away from 0;
  3. $f^k(N^-) \cap N = \varnothing$ for $0 \leq k < \ell$, $\lambda^\ell > 2$;

Theorem (Pesin '92)

In dimension 2, these assumptions imply the recurrence hypotheses of previous theorem.

- These three assumptions do not give existence in dimensions higher than 2. However, if hold for a higher-dimensional singular hyperbolic map, these assumptions still imply that there are *at most* finitely many SRBs, and the proof avoids using recurrence assumptions.

Singular hyperbolic flows

- Let M be a 3-dimensional Riemannian manifold, X a C^r vector field with flow X_t.
- A compact X_t-invariant subset $\Lambda \subset M$ is *singular hyperbolic* if
  1. Λ is *partially hyperbolic*, in that Λ admits an invariant splitting $T_\Lambda M = E^s \oplus E^c$ for which $d(X_t)|_{E^s}$ is contracting, E^s dominates E^c, and E^c is volume-expanding: there are $K > 0$ and $\lambda > 0$ for which:

$$\begin{aligned} \|d(X_t)|_{E^s}\| &\leq Ke^{-\lambda t}; \\ \|d(X_t)|_{E^s}\| \cdot \|d(X_t)|_{E^c}\| &\leq Ke^{-\lambda t}; \\ |J(d(X_t)|_{E^c})| &\geq Ke^{\lambda t} \quad \text{(where } J \text{ is the Jacobian)} \end{aligned}$$

  2. all singularities of X contained in Λ are hyperbolic (here "singularities" just means fixed points of the flow, or points $z \in M$ where $X(z) = 0$).
- The standard Lorenz system is an example of such a flow.

Spectral decomposition for flows

Theorem (Pacifico, Morales '07)

A singular hyperbolic attractor for a flow with dense periodic orbits and a unique singularity is a finite union of transitive sets. (If the vector field is "Kupka-Smale", this union is disjoint.)

- "*Kupka-Smale*" essentially means that the closed orbits and critical points are hyperbolic, and the stable manifold of a critical point can intersect the unstable manifold of another critical point only transversally.

Theorem (Araújo, Pacifico, Pujals, Viana '07)

A transitive attracting set Λ_i as above supports a unique SRB measure (physical probability measure which disintegrates to an absolutely continuous measure along center-unstable leaves).

Spectral decomposition for flows

- A consequence of these results is that a singular hyperbolic attractor Λ for a flow X on a 3-manifold M with dense periodic points is the following spectral decomposition:
  - $\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_k$, with Λ_i compact, X_t-invariant, and transitive;
  - There are exactly k ergodic SRB measures $\mu_1, \ldots, \mu_k$, with μ_i supported on Λ_i. Every SRB is a convex combination of the μ_i.

Theorem (Sataev '10)

If $\Lambda \subset M$ is a singular hyperbolic attractor for a flow X on a Riemannian n-manifold M whose stable distribution E^s has dimension $n-2$, then Λ admits a spectral decomposition as above.

Theorem (Araújo '22)

Let X be a vector field on a 3-manifold, Λ a singular hyperbolic attractor, s_L the number of hyperbolic singularities of X contained in Λ. Then the number s of ergodic SRB measures on Λ whose support contains a singularity is $s \leq 2s_L$.

Construction of SRB measures

- Define $W^u(z) = \bigcup_{n \geq 0} f^n(W^u_{\mathrm{loc}}(f^{-n}(z)))$ for $z \in K$ ($W^s(z)$ is defined analogously for $z \in \Lambda$).
- Let m^u_z and ρ^u_z be the Riemannian leaf volume and leaf metric on $W^u(z)$. Let $U_0 := B^u(z, r) \subset W^u_{\mathrm{loc}}(z)$ be the disc of ρ^u_z-radius r centered at z.
- Finally let $U_n = f(U_{n-1}) \setminus N^+$.
- Let $J^u(z) = \det\left(df|_{E^u(z)}\right)$ denote the unstable Jacobian of f at a point $z \in \Lambda$. For $y \in W^u(z)$, set

$$\kappa(z, y) = \prod_{j=0}^{\infty} \frac{J^u\left(f^{-j}(z)\right)}{J^u\left(f^{-j}(y)\right)}$$

Construction of SRB measures (cont.)

- Define the measures $\widetilde{\nu}_n$ on $U_n \subset W^u(f^n(z))$ by

$$d\widetilde{\nu}_n(y) = \widetilde{C}_n(z)\kappa(f^n(z), y)dm_z^u(y),$$

  where $\widetilde{C}_n(z)$ is a normalizing factor.
- Let ν_n be the measure on Λ given by $\nu_n(A) = \widetilde{\nu}_n(A \cap U_n)$ for Borel $A \subset \Lambda$.
- Define:

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \nu_k$$

- The final step in the construction is to show μ_n has an f-invariant weak limit measure μ concentrated on D. (Note μ may depend on the reference point z.)

Proof of finiteness: Preliminary constructions

- Recall $K^+ = \{x \in K : f^n(x) \notin N^+ \ \forall n \geq 0\}$ and $D = \bigcap_{n \geq 0} f^n(K^+)$.
- Given $x \in D$, in a neighborhood of size $\delta > 0$ of x, the unstable manifold $W^u(x)$ is a graph of a $C^{1+\alpha}$ function $E^u(x) \to E^s(x)$. Let $W^u_\delta(x)$ be the component of $W^u(x) \cap B_\delta(x) \cap D$ containing x.
- Given $\delta > 0$, let $B^-_\delta \subset D$ consist of those $x \in D$ for which $W^u_\delta(y)$ exists and contains x, for some $y \in D$.
- Suppose $\delta_1 < \delta_2$. Then $B^-_{\delta_2} \subseteq B^-_{\delta_1}$.
  - Indeed, if $x \in B^-_{\delta_2}$, then $x \in W^u_{\delta_2}(y)$ for some $y \in D$.
  - By certain regularity hypotheses, $D \cap W^u_{\delta_2}(y)$ has full measure, so can pick $y' \in W^u_{\delta_2}(y)$ that is with δ_1-distance to x.
  - Follows that $x \in B^-_{\delta_1}$.

Lemma

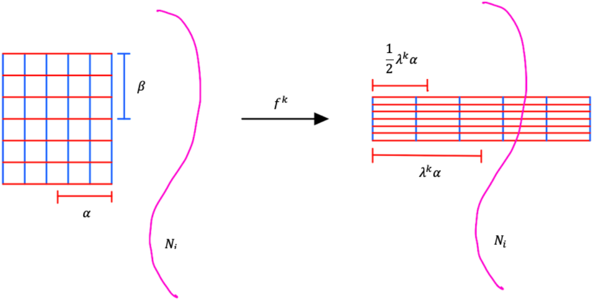
There exists a $\delta_0 > 0$ so that if μ is an ergodic SRB measure of $f : \Lambda \to \Lambda$, $\mu(B^-_{\delta_0}) > 0$.

Proving lemma

- B_δ^- is the set of points whose local unstable leaves have radius $\delta > 0$.
- Recall $N = \bigcup_{i=1}^m N_i$, and if U is a neighborhood of N, then $f(U)$ is a neighborhood of N^-.
- Since $f^k(N^-) \cap N = \varnothing$ for $1 \leq k < \ell$, $\lambda^\ell > 2$ ($\lambda > 1$ expansive constant), and N and $f^\ell(N^-)$ are closed, there is a radius $Q > 0$ so that:
    - the open neighborhoods $B_Q(N_i)$, $1 \leq i \leq m$, are disjoint;
    - $f^k(B_Q(N_i)) \cap N = \varnothing$ for $1 \leq k < \ell$.
- We let $\delta_0 < Q$.
- Choose ergodic SRB measure μ, and μ-generic point $x \in D$. Using hyperbolic product structure, construct rectangle R of stable leaves of radius $\beta > 0$ and unstable leaves of radius $\alpha > 0$. Then $\mu(R) > 0$.
- $f(R)$ is a rectangle whose unstable leaves have length $\lambda\alpha$.

Proving lemma (cont)

- **Idea:** Use iterates of R to extend the unstable leaves until they are of radius $\geq \delta_0$. Once this happens, $f^k(R) \subset B^-_{\delta_0}$, and $\mu(B^-_{\delta_0}) > \mu(f^k(R)) = \mu(R) > 0$.
- **Obstruction:** What if $f^k(R) \cap N \neq \varnothing$ before $\lambda^k \alpha > \delta_0$?
- Since $\delta_0 > Q$, $Q =$ radius of disjoint neighborhoods of components of N, $f^j(R)$ lies in one of these neighborhoods.
- Choose new rectangle R_1 of radius $\alpha_1 \geq \frac{1}{2}\lambda^k \alpha$.



Proving lemma (cont)

- Iterate R_1 until either:
  - $\lambda^k \alpha_1 \geq \delta_0$ (in which case $\mu(B^-_{\delta_0}) > 0$, and we're done); or
  - $f^{k_1}(R_1) \cap N \neq \varnothing$ for some $j_1 \geq \ell$ (since $R_1 \subset B_Q(N_i)$).
- In latter case, take new rectangle $R_2 \subset f^{k_1}(R_1) \setminus N$ with unstable leaves of radius $\alpha_2 \geq \frac{1}{2}\lambda^{k_1}\alpha_1$.
- Repeat this process. Each time a rectangle intersects N, we take a leaf of at least half the radius, creating a sequence of rectangles $\{R_m\}$ with unstable leaves of radii

$$\alpha_m \geq \frac{1}{2^m}\lambda^{k_1+\cdots+k_m}\alpha_1 > \frac{\lambda^{\ell m}}{2^m}\alpha_1 = \left(\frac{\lambda^\ell}{2}\right)^m \alpha_1,$$

  with each $k_m \geq \ell$ the time it takes for R_m to intersect N.
- Since $\lambda^\ell > 2$, this will eventually exceed δ_0, at which point $0 < \mu(R_m) \leq \mu(B^-_{\delta_0})$. □

Proving finiteness: Hopf argument

- The main result is proven once we show $B^-_{\delta_0}$ is charged by at most finitely many ergodic SRB measures.
- Let $\Lambda^\pm \subset \Lambda$ be the points on which the limits

$$\varphi_\pm(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi|_\Lambda \left(f^{\pm k}(x) \right)$$

both exist for every $\varphi \in C^0(K)$. Then $\mu(\Lambda^\pm) = 1$ by Birkhoff ergodic theorem, w.r.t. any invariant μ.

- We have:

$$y \in W^s_\delta(x) \iff d(f^n(x), f^n(y)) \xrightarrow{n\to\infty} 0,$$

$$y \in W^u_\delta(x) \iff d(f^{-n}(x), f^{-n}(y)) \xrightarrow{n\to\infty} 0.$$

In particular, φ_+ is constant on stable leaves and φ_- is constant on unstable leaves, for all $\varphi \in C^0(M)$.

Proving finiteness: Hopf argument (cont)

- **Idea:** Construct a closed (in K) subset $\Lambda^0 \subset B^-_{\delta_0}$ for which:
  1. $\varphi_+ = \varphi_-$ Leb-a.e. on unstable manifolds $W^u_\delta(x)$, and
  2. $\Lambda^0 \subset B^-_{\delta_0}$ has full measure with respect to any invariant measure.
- Partition Λ^0 into equivalence classes on which φ_+ is constant (note each $W^u_\delta(x)$ is contained in an equivalence class).
- Use the *local product structure* of the local foliations W^s_δ and W^u_δ to show that each equivalence class is open.
- Since Λ^0 is compact, there may only be finitely many such equivalence classes.
- Any SRB measure charging Λ^0 is supported (in Λ^0) on a union of these equivalence classes.
- Since Λ^0 is of full measure with respect to any invariant measure, and each equivalence class supports exactly one *ergodic* SRB measure (recall Ergodic Theorem), there may be only finitely many ergodic SRBs.

Muito obrigado!

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