Research Statement

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1 Introduction and Overview of Smooth Ergodic Theory

My general area of research is dynamical systems and smooth ergodic theory. Modern questions in dynamical systems began with the study of differential equations, investigating solutions to physical systems such as Newton's Three-Body Problem and models of atmospheric convection such as the Lorenz system. Finding explicit solutions to these systems of differential equations is exceptionally difficult, so we instead study their probabilistic behavior. Even this is challenging given the immense complexity of these systems. Therefore, the goal of smooth ergodic theory and hyperbolic dynamics is to develop tools for studying the statistical properties of simpler "chaotic" systems, in the hopes that these tools may eventually be used to better understand more complex dynamical systems in the real world.

A (discrete) dynamical system is a pair (X, f), in which X is a phase space and $f : X \to X$ is a transformation law (such as a smooth function) that moves points around the set X. Smooth ergodic theory considers dynamical systems over subsets of smooth manifolds, treated as probability spaces. The simplest examples studied are *uniformly hyperbolic*, and include Anosov diffeomorphisms, Smale horseshoes, and related stochastic systems in non-smooth categories (such as subshifts of finite type). Although uniformly hyperbolic maps are easy to construct, there are few examples (up to an appropriate equivalence relation), and the tools used to study them would not work with more complex physical systems. One area of active research examines hyperbolic dynamical systems exhibiting more pathological behavior (such as nonuniformly and partially hyperbolic maps, and hyperbolic maps with singularities), which also more closely resemble physical dynamical systems of practical interest (such as Lagrangian flow in fluid mechanics).

In my research, I have studied the statistical properties of three classes of hyperbolic dynamical systems:

(1) almost-Anosov diffeomorphisms of the torus [17];

(2) smooth models of pseudo-Anosov homeomorphisms of surfaces of arbitrary genus [18]; and

(3) hyperbolic dynamical systems with singular sets in manifolds of arbitrary dimension [19].

In the first two classes, I have discovered a range of probability measures that are invariant under the transformation law. This range includes a unique measure of maximal entropy, which satisfies the Central Limit Theorem and has exponential decay of correlations (see section 2.3). I plan to continue studying nonuniformly hyperbolic surface diffeomorphisms, such as pseudo-Anosov maps and systems with both elliptic and hyperbolic behavior (see e.g. [5] and [15]), and see if other techniques can be applied to these maps to study additional statistical properties.

Working in the class of singular hyperbolic maps, I have shown that singular hyperbolic attractors admit finitely many ergodic components, and that under moderate regulatory conditions, the SRB measure is unique if and only if the map restricted to the attractor is topologically transitive. I plan to construct examples to demonstrate that the proposed regulatory conditions are generally necessary for this result to hold, and further investigate the statistical properties of these maps, such as decay of correlations and the Central Limit Theorem. I am also currently investigating the ergodic and topological properties of attractors arising from singular hyperbolic maps with criticalities (points where the differential is not expanding), and developing novel techniques needed to study the hyperbolicity of such maps.

2 Thermodynamics of Hyperbolic Systems

The primary tools used in my research in smooth ergodic theory come from thermodynamic formalism. Thermodynamic formalism uses the principles of statistical physics as a framework to study the ergodic properties of abstract dynamical systems and stochastic processes. In this section, I will give a brief overview of the questions investigated in smooth ergodic theory, and the ways in which thermodynamic formalism has been used to investigate the ergodic theory of hyperbolic systems.

2.1 Historical motivation

The most basic class of stochastic transformations in the smooth category are *hyperbolic maps* $f : K \to M$, where $K \subset M$ is a precompact subset of a Riemannian manifold. An invertible map f is *uniformly hyperbolic* if there is a continuous splitting of the tangent space of $x \in K$ into two subspaces, $T_xM = E^u(x) \oplus E^s(x)$, for which we have $\lambda > 1$ such that:

$$|df_x^n v| \leq \lambda^{-n} |v| \quad \text{for every } v \in E^s(x); \quad \text{and} \tag{1}$$

$$\left|df_x^{-n} v\right| \leq \lambda^{-n} |v| \quad \text{for every } v \in E^u(x). \tag{2}$$

These maps include the *Anosov diffeomorphisms*, for which $K = M$. Given a continuous function $\varphi : K \to \mathbb{R}$ (called an *observable*) and a Borel probability measure μ on M, the sequence of functions $(\varphi \circ f^n)_{n\geq 0}$ on M forms a stochastic process.

Among the most significant probability measures studied are the *physical measures*, which are measures μ for which we have

$$m\left\{x \in M : \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(\varphi \circ f^k\right)(x) = \int \varphi \, d\mu\right\} > 0, \tag{3}$$

where m is the Lebesgue volume. The term "physical measure" comes from interpreting the sum as the average of a sequence of experimental observations of φ (the *time average*), and the integral as the expectation of φ (the *space average*).

In the 1970s, R. Bowen, D. Ruelle, and Y. Sinai described a large class of physical measures on uniformly hyperbolic dynamical systems. Their approach involved considering specific observables $\varphi \in C^0(M)$, and the *topological pressure* of the observable, defined by

$$P_f(\varphi) = \sup_{\mu\in\mathcal{M}_f} \left(h_\mu(f) + \int \varphi \, d\mu\right), \tag{4}$$

where $\mathcal{M}_f$ is the space of f-invariant Borel probability measures, and $h_\mu(f)$ is the metric entropy of f with respect to the measure μ. The sum $h_\mu(f) + \int \varphi \, d\mu$ is called *Helmholtz free energy* in statistical mechanics, and a probability distribution in a physical system that minimizes Helmholtz free energy is known as an *equilibrium state*[1].

Accordingly, in smooth ergodic theory, an *equilibrium state* is an f-invariant Borel probability measure μ that attains the supremum in (4). For certain choices of observables φ, the resulting equilibrium state is a physical measure. Therefore, one objective of thermodynamic formalism is to determine the existence and uniqueness of equilibrium states for a given dynamical system.

2.2 Classic examples of equilibrium states

In hyperbolic dynamics, at least two distinct equilibrium states for different observables are often discussed. For the observable $\varphi_0 \equiv 0$, an equilibrium state for φ_0 is a measure of maximal entropy. On the other hand, the observable $\varphi_1(x) = -\log \det \left| df_x|_{E^u(x)} \right|$ (known as the *geometric potential*) admits an equilibrium state known as a *Sinai-Ruelle-Bowen (SRB) measure*. SRB measures are physical measures that have absolutely continuous conditional measures on unstable submanifolds (which are the leaves of the foliation generated by the distribution E^u). For globally hyperbolic diffeomorphisms that preserve Riemannian volume, this volume is an SRB measure. On the other hand, for dissipative hyperbolic systems with attractors (see Section 3.3), the Riemannian volume of the attractor is 0, so SRB measures are better suited to use as physical measures supported on the state space.

Problem 1. Let $f : K \to M$ be a hyperbolic dynamical system.

(a) Does there exist a unique measure of maximal entropy for f?

(b) Does there exist a unique SRB measure for f?

For uniformly hyperbolic diffemorphisms, such as Anosov and Axiom A diffeomorphisms, it is known that the answer to both of these questions is "yes" [3]. However, for several classes of dynamical systems exhibiting more general types of hyperbolic behavior (such as nonuniformly hyperbolic systems and singular hyperbolic systems), the existence and uniqueness of SRB measures and measures of maximal entropy is an open problem.

2.3 Statistical questions

Once a probability measure μ for the dynamical system $f : K \to M$ has been chosen, a natural next step is to investigate the probability-theoretic properties of f. For example, a measurable dynamical system (X, f, μ) is said to satisfy the *Central Limit Theorem* with respect to a class of functions $\mathcal{H}$ on X (i.e., a class of random variables) if for any $h \in \mathcal{H}$ that is not a coboundary (i.e., $h \neq n' \circ f - h'$ for any $h' \in \mathcal{H}$), there exists $\sigma > 0$ such that

$$\lim_{n\to+\infty} \mu \left\{ \sqrt{n} \left(\frac{1}{n} \sum_{i=0}^{n-1} \left(h \circ f^i \right)(x) - \int h \, d\mu \right) < t \right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\tau^2/2\sigma^2} \, d\tau$$

[1]Helmholtz free energy in physics is in fact a negative multiple of this sum. This is why in statistical physics an equilibrium state *minimizes* Helmholtz free energy, whereas in thermodynamic formalism, an equilibrium measure *maximizes* this sum.

(i.e., the differences between the sample averages and expectation are asymptotically normally distributed with mean 0). Additionally, one may consider the *statistical correlations* $\int (h_1 \circ f^n)\, h_2\, d\mu - \int h_1\, d\mu \int h_2\, d\mu$ between a random variable h_2 and a stochastic process $h_1 \circ f^n$ for $h_1, h_2 \in \mathcal{H}$.

Problem 2. Suppose $f : K \to M$ is a hyperbolic dynamical system, and let μ be a physical measure on M.

(a) Is there a class of observables $\mathcal{H}$ on K for which f satisfies the central limit theorem?

(b) Given $h_1, h_2 \in \mathcal{H}$, do the correlations $\left|\int (h_1 \circ f^n)\, h_2\, d\mu - \int h_1\, d\mu \int h_2\, d\mu\right|$ between the process $h_1 \circ f^n$ and the random variable h_2 decay? If they do, how quickly?

3 Recent work

The thermodynamics and ergodic theory of uniformly hyperbolic diffeomorphisms have been thoroughly investigated. However, the smooth ergodic theory of both nonuniformly hyperbolic diffeomorphisms and singular hyperbolic attractors are rich areas of active research. My contribution to this body work has been to show existence and uniqueness of a range of equilibrium measures, including measures of maximal entropy, for smooth pseudo-Anosov diffeomorphisms and almost-Anosov diffeomorphisms of the torus; as well as show there are at most finitely many ergodic SRB measures for singular hyperbolic attractors of sufficient regularity.

3.1 Pseudo-Anosov smooth models

In [4], M. Gerber and A. Katok provided an explicit procedure to construct a nonuniformly hyperbolic C^∞ diffeomorphism $g : M \to M$ on any Riemannian surface M. The construction is a perturbation of pseudo-Anosov homeomorphisms, which are locally uniformly hyperbolic surface homeomorphisms that are smooth except at finitely many singular fixed points. In the construction, the trajectories near these singularities are slowed down, so that the differentials become the identity on the tangent space. Unlike Anosov diffeomorphisms, which only exist on the torus in two dimensions, pseudo-Anosov diffeomorphisms may be constructed on any compact surface (as every compact surface admits pseudo-Anosov homeomorphisms).

In [18], I investigated the existence and uniqueness of equilibrium states for a family of observables for pseudo-Anosov diffeomorphisms. The observables I studied were the one-parameter family of *geometric t-potentials* defined by $\varphi_t(x) = -t \log \det \left|dg_x|_{E^u(x)}\right|$. In particular, the equilibrium states for φ_0 and φ_1 are measures of maximal entropy and SRB measures, respectively. I was able to prove the following result, addressing Problems 1 and 2 for a range of equilibrium states for pseudo-Anosov diffeomorphisms.

Theorem 1. *Let M be a compact Riemannian 2-manifold. Given any $t_0 < 0$, there is a pseudo-Anosov diffeomorphism $g : M \to M$ for which the following hold:*

1. *For any $t \in (t_0, 1)$, there is a unique equilibrium measure μ_t associated to φ_t. This equilibrium measure has exponential decay of correlations and satisfies the Central Limit Theorem with respect to a class of functions containing all Hölder continuous functions*

on M, and is Bernoulli. Additionally, the pressure function $t \mapsto P_g(\varphi_t)$ is real analytic in the open interval $(t_0, 1)$.

2. *For $t = 1$, there are two classes of equilibrium measures associated to φ_1: convex combinations of Dirac measures concentrated at the singularities, and a unique SRB measure (which has density with respect to Riemannian volume).*

3. *For $t > 1$, the equilibrium measures associated to φ_t are precisely the convex combinations of Dirac measures concentrated at the singularities.*

My proof relies on the technology of Young towers, which are an adaptation of renewal theory to smooth ergodic theory. The thermodynamics and ergodic properties of these dynamical towers are an active area of research [13, 16], and are central to my work.

3.2 Almost-Anosov maps

Almost-Anosov diffeomorphisms were first described in [8], where H. Hu and L-S. Young investigated whether maps that are uniformly hyperbolic "essentially everywhere" admit SRB measures (addressing Problem 1). The answer to this question, as it turns out, is "not necessarily". Hu and Young constructed a diffeomorphism of the torus with uniform contraction along the stable leaves of the foliation, but with expansion that slowed down to 1 at a fixed point; in other words, the differential at the singular fixed point admitted eigenvalues of 1 and $\lambda < 1$. This "almost-Anosov diffeomorphism" does not admit an SRB probability measure, but it does admit an "infinite SRB measure", or an invariant measure with positive Lyapunov exponents almost everywhere, absolutely continuous conditional measures on the foliation, and giving finite mass to $M \setminus U$ for every neighborhood U of the singular fixed point.

Then in [7], Hu gave the following definition of an almost-Anosov map:

Definition. A map $f \in \mathrm{Diff}^4(M)$ is *almost-Anosov* if there exist two families of cones $\mathcal{C}^u_x \subset T_xM$ and $\mathcal{C}^s_x \subset T_xM$ (with $x \mapsto \mathcal{C}^u_x$ and $x \mapsto \mathcal{C}^s_x$ continuous) such that, except at a finite set of fixed points,

- $df_x\mathcal{C}^u_x \subset \mathcal{C}^u_{f(x)}$ and $df_x\mathcal{C}^s_x \supset \mathcal{C}^s_{f(x)}$, and
- $|df_xv| > |v|$ for $v \in \mathcal{C}^u_x$, and $|df_xv| < |v|$ for $v \in \mathcal{C}^s_x$.

Hu considered almost-Anosov maps whose differential slowed down to the identity at a single fixed point, known as an *indifferent* fixed point. His work showed that even when both expansion and contraction slow down to zero, there are examples of almost-Anosov maps admitting an SRB probability measure, and examples admitting an infinite SRB measure.

In [17], using thermodynamics of Young towers, I proved a similar result to Theorem 1:

Theorem 2. *Given an almost-Anosov diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ of the torus with an indifferent fixed point at the origin, there is a $t_0 < 0$ so that for any $t \in (t_0, 1)$, there is a unique equilibrium measure μ_t associated to $\varphi_t(x) = -t \log \det \left| df|_{E^u(x)} \right|$. This equilibrium state has exponential decay of correlations and satisfies the central limit theorem with respect to a class of functions containing all Hölder continuous functions on $\mathbb{T}^2$. In particular, this range of equilibrium measures includes a unique measure of maximal entropy.*

3.3 Singular hyperbolic attractors

In addition to my work in thermodynamics of nonuniformly hyperbolic surface diffeomorphisms, I have also studied the ergodic properties of singular hyperbolic dynamical systems. These are dynamical systems whose underlying state space $K \subset M$ contains a set of singularities $N \subset K$ where the map $f : K \setminus N \to K$ fails to be continuous and/or fails to be differentiable. A standard example of such a map is the geometric Lorenz attractor on the square $K = (-1,1)^2$, for which the function is discontinuous on the line $y = 0$; specifically the continuous extensions of this function to the rectangles $[-1,1]\times[-1,0]$ and $[-1,1]\times[0,1]$ each collapse this line to a single point (see subsection 4.1 for a more detailed description of the geometric Lorenz attractor.) The Lozi attractor is another such example, in which the function f is continuous on the singular set, but not differentiable.

In [11] it is shown that in general, singular hyperbolic maps admit SRB measures, but these measures may not be unique. In [19], I used a Hopf argument to show that if the singular set N is a disjoint union of finitely many closed submanifolds with boundary, then with certain other regulatory conditions, the singular hyperbolic attractor admits at most finitely many SRB measures. The hypothesis that N consists of finitely many closed submanifolds is easy to verify for most examples of singular hyperbolic attractors. I was also able to provide conditions under which the SRB measure is unique.

Theorem 3. *Let Λ be a singular hyperbolic attractor of a map $f : K \setminus N \to K$. Suppose the singular set N is a disjoint union of finitely many closed submanifolds with boundary.*

(a) Under certain regularity conditions on the singular set N and the map f, the restricted map $f|_\Lambda : \Lambda \to \Lambda$ admits finitely many ergodic components, and Λ admits finitely many distinct ergodic SRB measures.

(b) If the stable foliation of f in K satisfies certain regulatory conditions, then the number of distinct ergodic SRB measures of f correspond to the components of topological transitivity in Λ. In particular, under these assumptions, $f|_\Lambda : \Lambda \to \Lambda$ is topologically transitive if and only if Λ admits exactly one ergodic SRB measure.

4 Current work

4.1 Lorenz-type maps without invariant foliations

The geometric Lorenz attractor arises as the Poincaré first return map of the classical Lorenz system of differential equations:

$$\begin{cases} \dot{x} &= \sigma(y-x), \\ \dot{y} &= Rx - y - xz, \\ \dot{z} &= xy - \beta z. \end{cases} \tag{5}$$

Using the parameters $\sigma = 10$, $\beta = 8/3$, and $R \approx 28$, the first return map Φ has a line of discontinuity through the origin. Furthermore, the differential $d\Phi$ is unbounded near the singular line. For $R \approx 32$, the situation is more pathological: the attractor develops folds, leading to critical points of the map and tangencies between the stable and unstable submanifolds at certain points in the Poincaré return map.

One of my current projects, in collaboration with S. Luzzatto, M. Viana, and K. War, is to investigate the ergodic theory of Lorenz-type vector fields admitting critical trajectories similar to the Lorenz system (5) for $R \approx 32$. In fact, our setting is more general: we consider a one-parameter family of C^3 vector fields $(X_a)_{a\geq 0}$ on $\mathbb{R}^3$ with a hyperbolic singularity at the origin having one contracting and two expanding directions, and whose first-return maps to a small neighborhood in the square $\{(x,y,z) : z = 1,\ |x|, |y| \leq a\}$ satisfy certain geometric properties. This is a one-parameter family of *Lorenz-like vector fields,* and includes the Lorenz system with $\sigma = 10$ and $\beta = 8/3$, parametrized for $R > 0$.

Project 1. Given a one-parameter family of Lorenz-like vector fields $(X_a)_{a\geq 0}$, is there a set Ω of parameters a of positive Lebesgue measure for which X_a admits a nontrivial attractor with a saddle singularity and an infinite number of critical trajectories (i.e. trajectories with non-transversal intersection between stable and unstable leaves)?

In exploring this project, we've had to develop various technical strategies that are largely absent from the literature, such as variation and convergence of *hyperbolic coordinates*, and curvature and distortion estimates of curves under the dynamics. Although we began examining these techniques for the purpose of addressing Project 1, their setting is far more general, and so these techniques are more broadly applicable in hyperbolic dynamics.

Hyperbolic coordinates are a sequence of orthonormal coordinate frames $\{e^{(k)}, f^{(k)}\}_{k\geq 1}$ of the tangent bundle TM of a Riemannian surface M, corresponding to the singular value decomposition of the differential $d\Phi_x : T_xM \to T_{\Phi(x)}M$. In other words, at each point $x \in K$ in a hyperbolic set $K \subset M$, for each $k \geq 1$, $\{e^{(k)}_x, f^{(k)}_x\}$ is an orthonormal basis of T_xM for which $e^{(k)}_x$ (resp. $f^{(k)}_x$) minimizes (resp. maximizes) the magnitude $\|d\Phi^k_x v\|$ over unit vectors $v \in T_xM$. Hyperbolic coordinates were introduced implicitly in the work of M. Benedicks and L. Carleson in their exploration of the Hénon map [2]. Following their strategy, hyperbolic coordinates are important in our work on the topological structure of attractors for Lorenz-type maps with criticalities. Additionally, hyperbolic coordinates provide a very general framework for studying weak forms of hyperbolicity, so exploring the convergence and variation of hyperbolic coordinates to advance this framework is of independent interest.

Subproject 1.1. Let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism, and let $\{e^{(k)}_x, f^{(k)}_x\}_{k\geq 1}$ be a sequence of frames corresponding to the hyperbolic coordinates of iterates of Φ.

- What can we say about the convergence of $f^{(k)}$ as $k \to \infty$?
- How does $f^{(k)}_x$ vary as x varies?
- If Φ_a is a one-parameter family of diffeomorphisms, how does $f^{(k)}$ vary as a varies?

In addition to regularity of the hyperbolic coordinates, our results also require a bound on the *distortion* $\log\left|\det(d\Phi^n_x|_{E^u(x)})/\det(d\Phi^n_y|_{E^u(y)})\right|$ at finite time scales in n (known as *hyperbolic times* in the literature; see [1]). While the tangencies of the stable and unstable submanifolds can cause very slow expansion, the unbounded derivative near the singular set can cause arbitrarily large expansion. Together, these phenomena can lead to unbounded distortion at large time scales. One productive strategy has been to examine how the dynamics affect the curvature and distortion of smooth curves in the state space; this often leads to more geometrically intuitive computations than those involving only the differential of the map at different points.

As with the setting of Project 1.1, the setting in which we consider the curvature and distortion estimates of smooth curves under the dynamics is very general. Since distortion estimates form a significant computational obstacle for many investigations in hyperbolic dynamics, strong distortion estimates in a general setting with weak hyperbolicity would be very useful outside of our investigation into critical Lorenz systems.

Subproject 1.2. Suppose $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a C^2 diffeomorphism for which there is a *horizontal invariant cone field* $x \mapsto \mathcal{C}_x$ defined by

$$\mathcal{C}_x = \{(\omega, \sigma) \in T_x\mathbb{R}^2 : |\omega| \geq |\sigma|\}$$

What conditions on Φ give us suitable estimates of distortion in terms of known quantities such as $\|d\Phi^k_{\beta_i}\|$ and $\det(d\Phi^k_{\beta_i})$, for an orbit (β_i) and $i, k \geq 0$?

4.2 Further statistical properties of pseudo-Anosov maps

Having established the existence and uniqueness of SRB measures and measures of maximal entropy for smooth models of pseudo-Anosov diffeomorphisms in [18], and having proven both exponential decay of correlations and the Central Limit Theorem for the measure of maximal entropy, I am also working on further developing the statistical properties of smooth pseudo-Anosov maps with respect to their SRB measure. Additionally, for a pseudo-Anosov diffeomorphism $g : M \to M$, the one-parameter family of geometric t-potentials $\varphi_t(x) = -t \log \left| dg_x|_{E^u(x)} \right|$ only admits a unique equilibrium state for $t \in (t_0, 1)$ for some $t_0 < 0$ (not including convex combinations of Dirac point distributions); at $t = t_0$, the system exhibits a phase transition.

Project 2. Given a pseudo-Anosov homeomorphism $f : M \to M$ and a smooth realization $g : M \to M$ (i.e., a nonuniformly hyperbolic diffeomorphism g that is a perturbation of the original map f), what else can we say about the thermodynamic formalism of the map g? In particular:

- What are the statistical properties of the map g with respect to its SRB measure (such as decay of correlations, the Central Limit Theorem, and others)?
- What other classes of potential functions $\varphi : M \to \mathbb{R}$ admit unique equilibrium states?
- What are the statistical properties corresponding to these equilibrium states?

In [12], Y. Pesin, S. Senti, and F. Shahidi showed that the Lebesgue measure μ_1 for the Katok map of the torus has polynomial decay of correlations. The Katok map is an early example of a nonuniformly hyperbolic surface diffeomorphism, originally described in [10], and its construction is similar to the construction of the pseudo-Anosov diffeomorphisms described in [4]. For this reason, it seems likely that pseudo-Anosov diffeomorphisms also admit polynomial decay of correlations with respect to their invariant Riemannian area.

Subproject 2.1. Given a compact Riemannian manifold M, prove the existence of a smooth realization $g : M \to M$ of a pseudo-Anosov map admitting a unique SRB measure with polynomial decay of correlations.

Additionally, my current results on pseudo-Anosov maps only yield a range of equilibrium states up to a fixed $t_0 < 0$. In [20], T. Wang showed that the Katok map $G : \mathbb{T}^2 \to \mathbb{T}^2$ admits a unique equilibrium state μ_t for $\varphi_t(x) = -t \log \det \left| dG|_{E^u(x)} \right|$ for *every* $t < 0$. Furthermore, Wang also shows that μ_t has the *large deviations principle*, which estimates the rate of decay as $n \to \infty$ of the μ_t-measure of the set of points

$$L_n(\delta) := \left\{ x \in M : \left| \frac{1}{n} \sum_{k=0}^{n-1} h(f^k(x)) - \int h \, d\mu_t \right| > \delta \quad \forall h \in C^0(M) \right\},$$

for a given $\delta > 0$.

Subproject 2.2. Let $g : M \to M$ be a smooth realization of a pseudo-Anosov map.

- Does there exist a unique equilibrium state μ_t for $\varphi_t(x) = -t \log \det \left| dg|_{E^u(x)} \right|$ for any $t \in (-\infty, 1)$?
- What are the statistical properties of g with respect to μ_t?

The exact techniques in [12] and [20] cannot be applied directly to the pseudo-Anosov smooth realizations as defined in [4]; the slow-down procedure for the Katok map in [10] is generally speaking a more significant perturbation of a hyperbolic toral automorphism than the perturbation of a pseudo-Anosov homeomorphism to produce g. Therefore the technical arguments are notably different, and more investigation is required in order to study the statistical properties of these maps.

5 Future work: Other low-dimensional nonuniform hyperbolicity

I have also considered the thermodynamics of maps admitting both elliptic and hyperbolic behavior, such as the "elliptic island" systems described in [5] and [15].

Project 3. Under what conditions do maps admitting both elliptic and hyperbolic behavior also admit Young towers? What are the statistical properties of these maps with respect to SRB measures, measures of maximal entropy, and other equilibrium states?

Dynamical systems exhibiting both elliptic and hyperbolic behavior are famously difficult to analyze. A well-known example is the *Chirikov Standard Map* $f_k : \mathbb{T}^2 \to \mathbb{T}^2$, defined by

$$f_k(x, y) = (2x - y + k \sin(x), x).$$

Part of what makes analyzing the Standard Map and other "elliptic island" maps difficult is that unlike pseudo-Anosov diffeomorphisms, the angle between the subspaces $E^s(x)$ and $E^u(x)$ may become arbitrarily small, potentially obstructing nonuniform hyperbolicity. It is an open problem, for instance, to show that the Standard Map is nonuniformly hyperbolic for a Lebesgue-positive set of parameters k.

However, there are examples of toral maps that exhibit both elliptic and hyperbolic behavior, and are known to be nonuniformly hyperbolic. It is possible that the techniques used in [18] and [17] may be used to show that such maps admit SRB measures, measures of maximal entropy, and other equilibrium measures. This may provide insight into other ways of analyzing the Standard Map and other "elliptic island" systems.

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