SRB measures of singular hyperbolic attractors

Scuola Normale Superiore di Pisa

Dominic Veconi

International Centre for Theoretical Physics

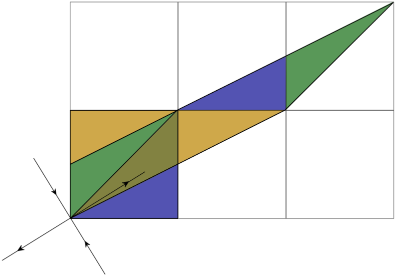
February 24 2022

Hyperbolic sets

- Let M be a Riemannian manifold, $U \subset M$ an open set, $f : U \to M$ a diffeomorphism onto its image.
- A compact f-invariant $\Lambda \subset M$ is a *hyperbolic set* if there exist constants $\lambda > 1$ and $c > 0$ such that each $x \in \Lambda$ admits a splitting $T_x M = E^s(x) \oplus E^u(x)$ for which
  - $(df_x)(E^s(x)) = E^s(f(x))$ and $(df_x)(E^u(x)) = E^u(f(x))$;
  - $\|df_x^n(v)\| \le c\lambda^{-n}\|v\| \ \forall v \in E^s(x)$;
  - $\|df_x^{-n}(f)\| \le c\lambda^{-n}\|v\| \ \forall v \in E^u(x)$.
- In this setting, we say f is *uniformly hyperbolic* on Λ.
- here are foliations W^s and W^u of U so that $T_x W^s(x) = E^s(x)$ and $T_x W^u(x) = E^u(x)$, and for which there is a $C > 0$, $0 < \alpha < 1$ such that for $x \in U$,
  - $f(W^s(x)) = W^s(f(x))$ and $f(W^u(x)) = W^u(f(x))$;
  - $\rho(f^n(x), f^n(y)) \le C\alpha^n \rho(x, y) \ \forall y \in W^s(x)$;
  - $\rho(f^{-n}(x), f^{-n}(y)) \le C\alpha^n \rho(x, y) \ \forall y \in W^u(x)$.
- We call $E^s(x)$, $E^u(x)$ the *stable/unstable subspaces* and $W^s(x)$, $W^u(x)$ the *stable/unstable submanifolds* at $x \in U$.

Example: Anosov diffeomorphisms

- If $\Lambda = M$ is a hyperbolic set for f, then f is an *Anosov diffeomorphism*.
- Consider the linear automorphism $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$. Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be the induced map on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.
- A has two eigendirections corresponding to the eigenvalues $\lambda^{-1} < 1 < \lambda$. These affine eigenspaces at each point in $\mathbb{R}^2$ descend to stable/unstable manifolds in $\mathbb{T}^2$.

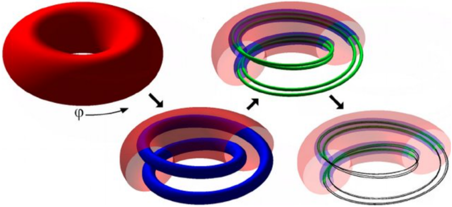


Example: Smale-Williams Solenoid

- Consider a map $F : \mathbb{S}^1 \times \mathbb{D} \to \mathbb{S}^1 \times \mathbb{D}$ of the solid torus given by

$$F(\varphi, x, y) = \left(2\varphi,\ \frac{1}{2}\cos(\varphi) + \frac{1}{5}x,\ \frac{1}{2}\sin(\varphi) + \frac{1}{5}y\right)$$

- Near the direction of $\mathbb{S}^1$, this map is expanding; in the direction of $\mathbb{D}$, the map is contracting.



- Notice: The toral automorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ preserved Lebesgue area. The solenoid map $F : \mathbb{S}^1 \times \mathbb{D} \to \mathbb{S}^1 \times \mathbb{D}$ does not preserve volume; in fact, the solenoid map is *dissipative* with respect to volume.

Singular hyperbolic attractors

Setting:

- M Riemannian manifold, $K \subset M$ open and precompact, $N \subset K$ closed, $N^+ = N \cup \partial K$;
- $f : K \setminus N \to K$ diffeomorphism onto its image;
- $N^- =$ image of continuous extensions of f to $N^+ \subset \overline{K}$; or more formally,

$$N^- = \left\{ y \in K : \exists z \in N^+, z_n \in K \setminus N \text{ s.t. } z_n \to z,\ f(z_n) \to y \right\}$$

- $K^+ = \{x \in K : f^n(x) \notin N^+ \ \forall n \geq 0\}$;
- $D = \bigcap_{n \geq 0} f^n(K^+)$, $\Lambda = \overline{D}$ (Λ is the *attractor* for f).
- Λ is a *singular hyperbolic attractor* if there is a continuous splitting $z \mapsto E^s(z) \oplus E^u(z)$ over $K \setminus N$ into *stable* and *unstable* subspaces. In particular, there are $C > 0$ and $\lambda > 1$ so that for any $z \in D$, $n \geq 0$:

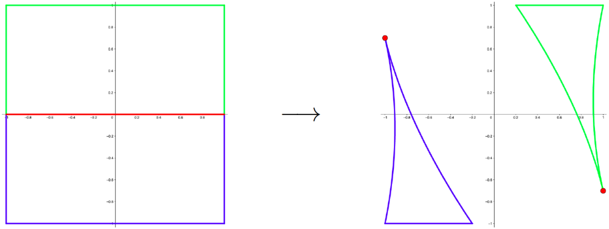
$$\begin{aligned} \left\| df_z^n v \right\| &\leq C\lambda^{-n} \left\| v \right\| \quad \forall v \in E^s(z); \\ \left\| df_z^{-n} v \right\| &\leq C\lambda^{-n} \left\| v \right\| \quad \forall v \in E^u(z). \end{aligned}$$

Example 1: Geometric Lorenz attractor

- $I = (-1,1)$, $K = I \times I$, $N = I \times \{0\}$, $f : K \setminus N \to K$ given by $f(x,y) = \big(\varphi(x,y), \psi(x,y)\big)$, where

$$\varphi(x,y) = (\operatorname{sgn}(y)Bx|y|^{\nu} - B|y|^{\nu_0} + 1)\operatorname{sgn}(y)$$
$$\psi(x,y) = ((1+A)|y|^{\nu_0} - A)\operatorname{sgn}(y)$$

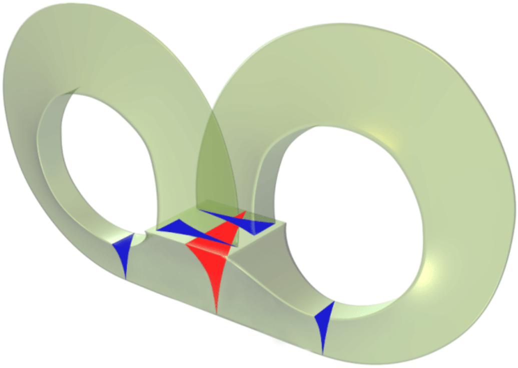
where $0 < A < 1$, $0 < B < \frac{1}{2}$, $\nu > 1$, and $1/(1+A) < \nu_0 < 1$.



- The two dots form the set N^-, the “image” of the singular set N.

Example: Geometric Lorenz attractor

The geometric Lorenz attractor is a Poincaré first-return map for a transverse cross-section of the flow of the classical Lorenz attractor.



Example: Lorenz-type maps

More generally, a *Lorenz-type map* is a map $f : K \setminus N \to K$, $K = I \times I$, $N = I \times \{a_0, a_1, \dots, a_{q+1}\}$, where

$$-1 = a_0 < a_1 < \cdots < a_q < a_{q+1} = 1$$

and, in addition to certain regularity conditions on df,

- $\lim_{y \uparrow a_i} f(x, y) = f_i^-$, $\lim_{y \downarrow a_i} f(x, y) = f_i^+$ ($f_i^\pm \in \overline{K \setminus N}$ constant points, independent of $x \in I$);
- $f|_{I \times (a_i, a_{i+1})} : I \times (a_i, a_{i+1}) \to K$ is a diffeomorphism onto its image;
- $\|\varphi_x\|, \left\|\psi_y^{-1}\right\| < 1$, where $f(x, y) = (\varphi(x, y), \psi(x, y))$.

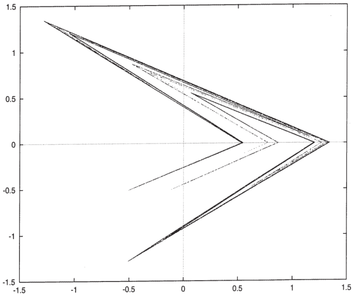
Theorem (Afraimovich, Bykov, Shilnikov '83)

If M is a compact Riemannian manifold w/ $\dim M \geq 3$, there exists a vector field X and a smooth submanifold S such that the first-return time map f induced on S by the flow given by X is a Lorenz-type map.

Example: Lozi attractor

- Lozi map: A simplified Hénon map $f : K \setminus N \to K$, $K = (-c, c)^2$, $c \in (0, 1.5)$, $N = \{0\} \times (-c, c)$, $a > 0$, $b > 0$ sufficiently small:

$$f(x, y) = (1 + by - a|x|, x)$$



- In this case, the map is continuous on N, but not differentiable.

SRB measures

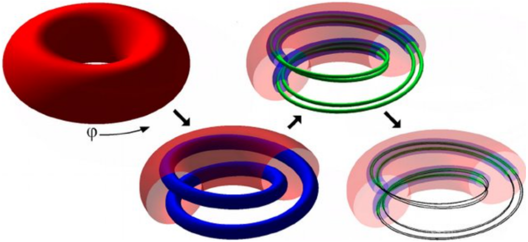
- Suppose $f : U \to M$ is a hyperbolic map on a Riemannian manifold M. An *SRB measure* is an invariant Borel probability measure μ for which:
  - f has positive Lyapunov exponents μ-a.e., and
  - μ admits absolutely continuous conditional measures on the unstable leaves $W^u(x)$ (w.r.t. Riemannian leaf volume)
- SRB measures are hyperbolic *physical measures*: $m(\mathcal{B}_\mu) > 0$, where m is the Lebesgue/Riemannian volume and $\mathcal{B}_\mu$ is the *basin* of μ:

$$\mathcal{B}_\mu := \left\{ x : \frac{1}{n} \sum_{k=0}^{n-1} \left(\varphi \circ f^k \right)(x) \xrightarrow{n \to \infty} \int_U \varphi \, d\mu \quad \forall \varphi \in C^0 \right\}$$

- In ergodic theory, invariant measures correspond to stationary distributions in probability theory. So SRB measures are *stationary distributions that satisfy the strong law of large numbers on a set of positive volume.*

Conservative and dissipative systems

- If $f : M \to M$ is an area-preserving diffeomorphism (e.g. Anosov map), then the Lebesgue/Riemannian volume is an SRB measure.
- What if f is dissipative? Recall the solenoid map:



- The attractor $\Omega = \bigcap_{n\geq 1} F^n(\mathbb{S}^1 \times \mathbb{D})$ is locally a product of an interval and a Cantor set. In particular, it has Lebesgue measure 0.
- The SRB measure is a product of normalized Lebesgue measure on $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{R}$ and the Bernoulli measure on the Cantor set.

SRB measures for hyperbolic maps

Theorem (Sinai '72, Bowen, Ruelle '75, Ruelle '76)

Suppose $f : U \to M$ is $C^{1+\alpha}$ and $\Lambda = \bigcap_{n \geq 0} f^n(U)$ is uniformly hyperbolic. Then there are at most finitely many ergodic SRB measures on Λ. If, furthermore, $f|_\Lambda$ is topologically transitive, then there is a unique SRB measure μ on Λ, and $\mathcal{B}_\mu$ has full measure in U.

Theorem (Rodriguez-Hertz, Rodriguez-Hertz, Tahzibi, Ures '10)

If $f : M \to M$ is a topologically transitive $C^{1+\alpha}$ diffeomorphism, then it admits at most one SRB measure.

Theorem (Pesin '92)

Suppose $f : K \setminus N \to K$ admits a singular hyperbolic attractor Λ. Then there are at most countably many ergodic SRB measures supported on Λ.

Stable and unstable manifolds

- We use *unstable manifolds* to construct SRB measures.
- Assume $f : K \setminus N \to K$ is singular hyperbolic with attractor Λ, and let ρ denote the Riemannian distance in $M \supset K$, m the Riemannian measure.

Theorem (Pesin '92)

For m-a.e. every $z \in K \setminus N$, there are embedded submanifolds $W^s_{\mathrm{loc}}(z)$ and $W^u_{\mathrm{loc}}(z)$ containing z for which $T_z W^s_{\mathrm{loc}}(z) = E^s(z)$ and $T_z W^u_{\mathrm{loc}}(z) = E^u(z)$. Furthermore, there is an $\alpha < 1$ and $C > 0$ for which, for all $n \geq 0$, letting ρ denote Riemannian distance,

$$\rho(f^n(x), f^n(y)) \leq C\alpha^n \rho(x,y) \quad \text{for } x, y \in W^s_{\mathrm{loc}}(z),$$
$$\rho(f^{-n}, f^{-n}(y)) \leq C\alpha^n \rho(x,y) \quad \text{for } x, y \in W^u_{\mathrm{loc}}(z).$$

Additionally, for $w \in N \setminus K$ sufficiently close to z, the intersection $W^s_{\mathrm{loc}}(z) \cap W^u_{\mathrm{loc}}(w)$ is nonempty and contains exactly one point.

Construction of SRB measures

- Define $W^u(z) = \bigcup_{n\geq 0} f^n\left(W^u_{\mathrm{loc}}(f^{-n}(z))\right)$ for $z \in K$ ($W^s(z)$ is defined analogously for $z \in \Lambda$).
- Let $J^u(z) = \det\left(df|_{E^u(z)}\right)$ denote the unstable Jacobian of f at a point $z \in \Lambda$. For $y \in W^u(z)$, set

$$\kappa(z, y) = \prod_{j=0}^{\infty} \frac{J^u\left(f^{-j}(z)\right)}{J^u\left(f^{-j}(y)\right)}$$

- Let m^u_z and ρ^u_z be the Riemannian leaf volume and leaf metric on $W^u(z)$. Let $U_0 := B^u(z, r) \subset W^u_{\mathrm{loc}}(z)$ be the disc of ρ^u_z-radius r centered at z.
- Finally let $U_n = f(U_{n-1}) \setminus N^+$.

Construction of SRB measures (cont.)

- Define the measures $\widetilde{\nu}_n$ on $U_n \subset W^u(f^n(z))$ by

$$d\widetilde{\nu}_n(y) = \widetilde{C}_n(z)\kappa(f^n(z), y)dm_z^u(y),$$

  where $\widetilde{C}_n(z)$ is a normalizing factor.
- Let ν_n be the measure on Λ given by $\nu_n(A) = \widetilde{\nu}_n(A \cap U_n)$ for Borel $A \subset \Lambda$.
- Each ν_n is defined only on subsets of a particular unstable manifold $W^u(f^n(z))$.
- Define:

$$\mu_n = \frac{1}{n}\sum_{k=0}^{n-1} \nu_k$$

- The final step in the construction is to show μ_n has an f-invariant weak limit measure μ concentrated on D. (Note μ may depend on the reference point z.)

Setting of main result

Setting:

- M Riemannian manifold, $K \subset M$ open and precompact, $N \subset K$ closed;
- $f : K \setminus N \to K$ diffeomorphism onto its image;
- $N^- =$ image of continuous extensions of f to $N^+ \subset \overline{K}$; or more formally,

$$N^- = \left\{ y \in K : \exists z \in N^+, z_n \in K \setminus N \text{ s.t. } z_n \to z,\ f(z_n) \to y \right\}$$

  (for example, $N^- = \{f_i^\pm : 1 \leq i \leq q\}$ for Lorenz-type maps, where $f_i^+ = \lim_{y \downarrow a_i} f(x,y)$ and $f_i^- = \lim_{y \uparrow a_i} f(x,y)$);
- Λ a singular hyperbolic attractor, expansive constant $\lambda > 1$.

Main result

Assumptions:

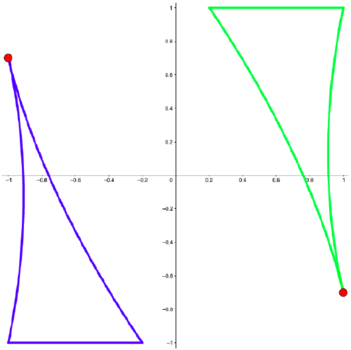
1. N is a disjoint union of finitely many closed submanifolds $N_1, \ldots, N_m$ with boundary, of dimension equal to the codimension of W^u;
2. The local unstable manifolds $W^u_{\mathrm{loc}}(x)$ intersect the singular set N transversally, with angle uniformly bounded away from 0;
3. $f^j(N^-) \cap N = \varnothing$ for $0 \leq j < k$, $\lambda^k > 2$.

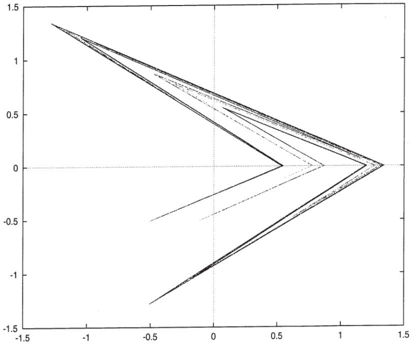
Theorem (V. 2022)

If $f : K \setminus N \to K$ as above satisfies these assumptions, then the attractor Λ admits finitely many ergodic SRB measures.

Lorenz and Lozi revisited

- This implies, in particular, that the attractors for both Lorenz-type maps and the Lozi map admit finitely many SRB measures. (Observe that $W^u_{\mathrm{loc}}(x)$ lies inside the attractor Λ for all $x \in \Lambda$ where $W^u_{\mathrm{loc}}(x)$ is defined.)





A simple non-example

- Assumption that N is a finite union of submanifolds is required for arguments. Result may not hold if N has infinitely many components.
- Example: Take a countable number of horizontal lines in $(-1, 1)^2$.
- In each resulting section, embed a copy of the geometric Lorenz attractor.
- Each section admits its own SRB measure.
- The proof of the main result requires a lower bound on the distance between components of N, which we don't have in this example.

Local continuity of stable foliation

- Is there a connection between the number/supports of SRB measures and the topological properties of the dynamics (e.g. topological transitivity/transitive components)?
- Given $z \in K \setminus N$, there is a radius $r = r(z) > 0$ for which $W^s_{\mathrm{loc}}(z) \cap B_r(z)$ is the graph of a C^1 function $\psi_z : U_z \to M$, $U_x z \subset T_z W^s_{\mathrm{loc}}(z)$.
- W^s is *locally continuous* if $y \mapsto \psi_y$ and $(y, u) \mapsto d(\psi_y)_u$ are continuous over $y \in (K \setminus N) \cap B_{r(x)}(x)$, $u \in U_y \subset T_y W^s(y)$. (Note ψ_y is defined μ-a.e. in $(K \setminus N) \cap B_{r(x)}(x)$, μ any SRB measure.)
- In particular, the maximal radius $z \mapsto r(z)$ in which $W^s_{\mathrm{loc}}(z) \cap B_{r(z)}(z)$ is a C^1 curve varies continuously over Λ.

Transitivity and ergodicity

Theorem (Pesin 1992, V. 2022)

Let $f : K \setminus N \to K$ be singular hyperbolic.

- *There is a countable collection of f-invariant subsets $\{U_i\}_{i\geq 1}$, open in Λ, for which $\overline{\bigcup_{i\geq 1} U_i} = \Lambda$, and each of which is supported by exactly one ergodic SRB measure.*
- *If N is a finite disjoint union of embedded submanifolds of dimension equal to the codimension of W^u, and if each unstable curve W^u intersects N transversally with angle uniformly bounded away from 0, then this collection is finite.*
- *If $f|_\Lambda : \Lambda \to \Lambda$ is topologically transitive, then $U_1 = \Lambda$ is the only member of this collection, and thus the ergodic SRB measure is unique.*

Construction of ergodic components

- The construction of the ergodic components U_i is due to Pesin '92.
- Let μ be an SRB measure for Λ. For μ-a.e. $z \in \Lambda$, $W^u_{\mathrm{loc}}(z) \subset \Lambda$ and the stable discs $B^s_{r(y)}(y)$ are defined for m^u_z-a.e. $y \in W^u_{\mathrm{loc}}(z)$, i.e. on a set $A^u(z) \subset W^u_{\mathrm{loc}}(z)$ of full m^u_z-measure. (Recall m^u_z is the Riemannian leaf volume of $W^u_{\mathrm{loc}}(z)$.)
- Define the set

$$Q(z) = \left(\bigcup_{y \in A^u(z)} B^s_{r(y)}(y) \right) \cap \Lambda \cap B_{r(z)}(z).$$

Note $\mu(Q(z)) > 0$.

- Note $Q = \bigcup_{n \in \mathbb{Z}} f^n(Q(z))$ is f-invariant, and thus an ergodic component of μ.
- Openness of Q (mod 0) in Λ follows from the local continuity of W^s (i.e. continuity of $y \mapsto r(y)$ for $y \in W^u_{\mathrm{loc}}(z)$).

Proof of finiteness: Preliminary constructions

- Recall $K^+ = \{x \in K : f^n(x) \notin N^+ \, \forall n \geq 0\}$ and $D = \bigcap_{n\geq 0} f^n(K^+)$.
- Given $\delta > 0$, let $B_\delta^- \subset D$ consist of those $x \in D$ for which $W_\delta^u(y)$ exists and contains x, for some $y \in D$.
- Suppose $\delta_1 < \delta_2$. Then $B_{\delta_2}^- \subseteq B_{\delta_1}^-$.
  - Indeed, if $x \in B_{\delta_2}^-$, then $x \in W_{\delta_2}^u(y)$ for some $y \in D$.
  - By certain regularity hypotheses, $D \cap W_{\delta_2}^u(y)$ has full measure, so can pick $y' \in W_{\delta_2}^u(y)$ that is with δ_1-distance to x.
  - Follows that $x \in B_{\delta_1}^-$.

Lemma

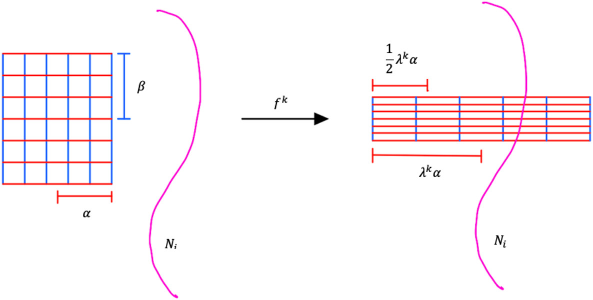
There exists a $\delta_0 > 0$ so that if μ is an ergodic SRB measure of $f : \Lambda \to \Lambda$, $\mu(B_{\delta_0}^-) > 0$.

Proving first lemma

- B_δ^- is the set of points whose local unstable leaves have radius $\delta > 0$.
- Recall $N = \bigcup_{i=1}^m N_i$, and if U is a neighborhood of N, then $f(U)$ is a neighborhood of N^-.
- Since $f^j(N^-) \cap N = \varnothing$ for $1 \leq j < k$, $\lambda^k > 2$ ($\lambda > 1$ expansive constant), and N and $f^k(N^-)$ are closed, there is a radius $Q > 0$ so that:
  - the open neighborhoods $B_Q(N_i)$, $1 \leq i \leq m$, are disjoint;
  - $f^j(B_Q(N_i)) \cap N = \varnothing$ for $1 \leq j < k$.
- We let $\delta_0 < Q$.
- Choose ergodic SRB measure μ, and μ-generic point $x \in D$. Using hyperbolic product structure, construct rectangle R of stable leaves of radius $\beta > 0$ and unstable leaves of radius $\alpha > 0$. Then $\mu(R) > 0$.
- $f(R)$ is a rectangle whose unstable leaves have length $\lambda\alpha$.

Proving first lemma (cont)

- **Idea:** Use iterates of R to extend the unstable leaves until they are of radius $\geq \delta_0$. Once this happens, $f^j(R) \subset B^-_{\delta_0}$, and $\mu(B^-_{\delta_0}) > \mu(f^j(R)) = \mu(R) > 0$.
- **Obstruction:** What if $f^j(R) \cap N \neq \varnothing$ before $\lambda^j \alpha > \delta_0$?
- Since $\delta_0 > Q$, $Q =$ radius of disjoint neighborhoods of components of N, $f^j(R)$ lies in one of these neighborhoods.
- Choose new rectangle R_1 of radius $\alpha_1 \geq \frac{1}{2}\lambda^j \alpha$.



Proving first lemma (cont)

- Iterate R_1 until either:
  - $\lambda^j \alpha_1 \geq \delta_0$ (in which case $\mu(B^-_{\delta_0}) > 0$, and we're done); or
  - $f^{j_1}(R_1) \cap N \neq \varnothing$ for some $j_1 \geq k$ (since $R_1 \subset B_Q(N_i)$).
- In latter case, take new rectangle $R_2 \subset f^{j_1}(R_1) \setminus N$ with unstable leaves of radius $\alpha_2 \geq \frac{1}{2}\lambda^{j_1}\alpha_1$.
- Repeat this process. Each time a rectangle intersects N, we take a leaf of at least half the radius, creating a sequence of rectangles $\{R_\ell\}$ with unstable leaves of radii

$$\alpha_\ell \geq \frac{1}{2^\ell}\lambda^{j_1+\cdots+j_\ell}\alpha_1 > \frac{\lambda^{k\ell}}{2^\ell}\alpha_1 = \left(\frac{\lambda^k}{2}\right)^l \alpha_1,$$

  with each $j_\ell \geq k$ the time it takes for R_ℓ to intersect N.
- Since $\lambda^k > 2$, this will eventually exceed δ_0, at which point $0 < \mu(R_\ell) \leq \mu(B^-_{\delta_0})$. □

Proving finiteness

- The main result is proven once we show $B^-_{\delta_0}$ is charged by at most finitely many ergodic SRB measures
- Let $\Lambda^0 \subset \Lambda$ be the points on which the limits

$$\varphi_\pm(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi|_\Lambda \left(f^{\pm k}(x) \right)$$

  both exist for every $\varphi \in C^0(K)$. Then $\mu(\Lambda^0) = 1$ by Birkhoff ergodic theorem, w.r.t. any invariant μ.
- Partition Λ^0 into equivalence classes on which φ_+ and φ_- are constant (and equal).
- These equivalence classes are clopen in Λ. Since Λ is compact, there are at most finitely many equivalence classes.
- Any ergodic SRB measure on Λ is supported on one of these equivalence classes, and each equivalence class can support at most one SRB measure.

References

- V. S. Afraimovich, V. V. Bykov, L. P. Shilnikov. On attracting nonstructurally stable limit sets of Lorenz attractor type, *Trudy Moskov. Mat. Obshch.* **44**:150-212, 1983.
- R. Bowen. Equilibrium states and the ergodic theory of Anosov diffeomorphisms. *Lecture Notes in Mathematics* **470**, Springer-Verlag, Berlin. 1975.
- E. Clay, B. Hasselblatt, E. Pujals. Desingularization of surface maps, *Electron. Res. Announc.* **24**:1-9, 2017.
- V. Climenhaga, S. Luzzatto, Y. Pesin. The geometric approach for constructing Sinai-Ruelle-Bowen measures (Survey). *J. Stat. Phys.* **166**:467-493, 2016.
- Y. Pesin. Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties. *Ergod. Th. & Dynam. Sys.* **12**:123-151, 1992.
- F. Rodriguez-Hertz, M. A. Rodriguez-Hertz, A. Tahzibi, R. Ures. Uniqueness of SRB measures for transitive diffeomorphisms on surfaces, *Commun. Math. Phys.*, **306**(1):35-49, 2011.
- D. Ruelle. A measure associated with axiom-A attractors. *Amer. J. Math* **98**(3):619-654, 1976.
- Y. Sinai. Markov partitions and Y-diffeomorphisms. *Funct. Anal. and Appl.* **2**(1):64-89, 1968.
- D. Veconi. SRB measures of singular hyperbolic attractors. To appear in *Discrete and Contin. Dyn. Sys.* 2022.