Research Statement

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1 Introduction and Overview of Smooth Ergodic Theory

My general area of research is dynamical systems and smooth ergodic theory. Modern questions in dynamical systems began with the study of differential equations, investigating solutions to physical systems such as Newton's Three-Body Problem and models of atmospheric convection such as the Lorenz system. Finding explicit solutions to these systems of differential equations is exceptionally difficult, so we instead study their probabilistic behavior. Even this is challenging given the immense complexity of these systems. Therefore, the goal of smooth ergodic theory and hyperbolic dynamics is to develop tools for studying the statistical properties of simpler "chaotic" systems, in the hopes that these tools may eventually be used to better understand more complex dynamical systems in the real world.

A (discrete) dynamical system is a pair (X, f), in which X is a phase space and $f : X \to X$ is a transformation law (such as a smooth function) that moves points around the set X. Smooth ergodic theory considers dynamical systems over subsets of smooth manifolds, treated as probability spaces. The simplest examples studied are *uniformly hyperbolic*, and include Anosov diffeomorphisms, Smale horseshoes, and related stochastic systems in non-smooth categories (such as subshifts of finite type). Although uniformly hyperbolic maps are easy to construct, there are few examples (up to an appropriate equivalence relation), and the tools used to study them would not work with more complex physical systems. One area of active research examines hyperbolic dynamical systems exhibiting more pathological behavior (such as nonuniformly and partially hyperbolic maps, and hyperbolic maps with singularities), which also more closely resemble physical dynamical systems of practical interest (such as Lagrangian flow in fluid mechanics).

In my research, I have studied the statistical properties of two classes of nonuniformly hyperbolic dynamical systems:

(1) almost-Anosov diffeomorphisms of the torus [14];

(2) smooth models of pseudo-Anosov homeomorphisms of surfaces of arbitrary genus [15]; and

(3) hyperbolic dynamical systems with singular sets in manifolds of arbitrary dimension [16].

In the first two classes, I have discovered a range of probability measures that are invariant under the transformation law. This range includes a unique measure of maximal entropy, which satisfies the Central Limit Theorem and has exponential decay of correlations (see section 2.3). I plan to continue studying nonuniformly hyperbolic surface diffeomorphisms, such as pseudo-Anosov maps and systems with both elliptic and hyperbolic behavior (see e.g. [3] and [12]), and see if other techniques can be applied to these maps to study additional statistical properties.

Working in the class of singular hyperbolic maps, I have shown that singular hyperbolic attractors admit finitely many ergodic components, and that under moderate regulatory conditions, the SRB measure is unique if and only if the map restricted to the attractor is topologically transitive. I plan to construct examples to demonstrate that the proposed regulatory conditions are generally necessary for this result to hold, and further investigate the statistical properties of these maps, such as decay of correlations and the Central Limit Theorem.

2 Thermodynamics of Hyperbolic Systems

The primary tools used in my research in smooth ergodic theory come from thermodynamic formalism. Thermodynamic formalism uses the principles of statistical physics as a framework to study the ergodic properties of abstract dynamical systems and stochastic processes. In this section, I will give a brief overview of the questions investigated in smooth ergodic theory, and the ways in which thermodynamic formalism has been used to investigate the ergodic theory of hyperbolic systems.

2.1 Historical motivation

The most basic class of stochastic transformations in the smooth category are *hyperbolic maps* $f : K \to M$, where $K \subset M$ is a precompact subset of a Riemannian manifold. An invertible map f is *uniformly hyperbolic* if there is a continuous splitting of the tangent space of $x \in K$ into two subspaces, $T_xM = E^u(x) \oplus E^s(x)$, for which we have $\lambda > 1$ such that:

$$|df_x^n v| \leq \lambda^{-n} |v| \quad \text{for every } v \in E^s(x); \quad \text{and} \tag{1}$$

$$\left|df_x^{-n} v\right| \leq \lambda^{-n} |v| \quad \text{for every } v \in E^u(x). \tag{2}$$

These maps include the *Anosov diffeomorphisms*, for which $K = M$. Given a continuous function $\varphi : K \to \mathbb{R}$ (called an *observable*) and a Borel probability measure μ on M, the sequence of functions $(\varphi \circ f^n)_{n\geq 0}$ on M forms a stochastic process.

Among the most significant probability measures studied are the *physical measures*, which are measures μ for which we have

$$m\left\{x \in M : \lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left(\varphi \circ f^k\right)(x) = \int \varphi \, d\mu\right\} > 0, \tag{3}$$

where m is the Lebesgue volume. The term “physical measure” comes from interpreting the sum as the average of a sequence of experimental observations of φ (the *time average*), and the integral as the expectation of φ (the *space average*).

In the 1970s, R. Bowen, D. Ruelle, and Y. Sinai described a large class of physical measures on uniformly hyperbolic dynamical systems. Their approach involved considering specific observables $\varphi \in C^0(M)$, and the *topological pressure* of the observable, defined by

$$P_f(\varphi) = \sup_{\mu\in\mathcal{M}_f} \left(h_\mu(f) + \int \varphi \, d\mu\right), \tag{4}$$

where $\mathcal{M}_f$ is the space of f-invariant Borel probability measures, and $h_\mu(f)$ is the metric entropy of f with respect to the measure μ. The sum $h_\mu(f) + \int \varphi \, d\mu$ is called *Helmholtz* *free energy* in statistical mechanics, and a probability distribution in a physical system that minimizes Helmholtz free energy is known as an *equilibrium state*[1].

Accordingly, in smooth ergodic theory, an *equilibrium state* is an f-invariant Borel probability measure μ that attains the supremum in (4). For certain choices of observables φ, the resulting equilibrium state is a physical measure. Therefore, one objective of thermodynamic formalism is to determine the existence and uniqueness of equilibrium states for a given dynamical system.

2.2 Classic examples of equilibrium states

In hyperbolic dynamics, at least two distinct equilibrium states for different observables are often discussed. For the observable $\varphi_0 \equiv 0$, an equilibrium state for φ_0 is a measure of maximal entropy. On the other hand, the observable $\varphi_1(x) = -\log\det\left|df_x|_{E^u(x)}\right|$ (known as the *geometric potential*) admits an equilibrium state known as a *Sinai-Ruelle-Bowen (SRB) measure*. SRB measures are physical measures that have absolutely continuous conditional measures on unstable submanifolds (which are the leaves of the foliation generated by the distribution E^u). For globally hyperbolic diffeomorphisms that preserve Riemannian volume, this volume is an SRB measure. On the other hand, for dissipative hyperbolic systems with attractors (see Section **??**), the Riemannian volume of the attractor is 0, so SRB measures are better suited to use as physical measures supported on the state space.

Problem 1. Let $f : K \to M$ be a hyperbolic dynamical system.

(a) Does there exist a unique measure of maximal entropy for f?

(b) Does there exist a unique SRB measure for f?

For uniformly hyperbolic diffemorphisms, such as Anosov and Axiom A diffeomorphisms, it is known that the answer to both of these questions is "yes" [1]. However, for several classes of dynamical systems exhibiting more general types of hyperbolic behavior (such as hyperbolic systems admitting singularities), the existence and uniqueness of SRB measures and measures of maximal entropy is an open problem.

2.3 Statistical questions

Once a probability measure μ for the dynamical system $f : K \to M$ has been chosen, a natural next step is to investigate the probability-theoretic properties of f. For example, a measurable dynamical system (X, f, μ) is said to satisfy the *Central Limit Theorem* with respect to a class of functions $\mathcal{H}$ on X (i.e., a class of random variables) if for any $h \in \mathcal{H}$ that is not a coboundary (i.e., $h \neq n' \circ f - h'$ for any $h' \in \mathcal{H}$), there exists $\sigma > 0$ such that

$$\lim_{n\to+\infty} \mu\left\{\sqrt{n}\left(\frac{1}{n}\sum_{i=0}^{n-1}\left(h\circ f^i\right)(x) - \int h\,d\mu\right) < t\right\} = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\tau^2/2\sigma^2}\,d\tau$$

[1] Helmholtz free energy in physics is in fact a negative multiple of this sum. This is why in statistical physics an equilibrium state *minimizes* Helmholtz free energy, whereas in thermodynamic formalism, an equilibrium measure *maximizes* this sum.

(i.e., the differences between the sample averages and expectation are asymptotically normally distributed with mean 0). Additionally, one may consider the *statistical correlations* $\int (h_1 \circ f^n)\, h_1\, d\mu - \int h_1\, d\mu \int h_2\, d\mu$ between a random variable h_2 and a stochastic process $h_1 \circ f^n$ for $h_1, h_2 \in \mathcal{H}$.

Problem 2. Suppose $f : K \to M$ is a hyperbolic dynamical system, and let μ be a physical measure on M.

(a) Is there a class of observables $\mathcal{H}$ on K for which f satisfies the central limit theorem?

(b) Given $h_1, h_2 \in \mathcal{H}$, do the correlations $\left|\int (h_1 \circ f^n)\, h_2\, d\mu - \int h_1\, d\mu \int h_2\, d\mu\right|$ between the process $h_1 \circ f^n$ and the random variable h_2 decay? If they do, how quickly?

3 Recent work: Nonuniformly hyperbolic surface diffeomorphisms

The thermodynamics and ergodic theory of uniformly hyperbolic diffeomorphisms have been thoroughly investigated. However, the smooth ergodic theory of nonuniformly hyperbolic diffeomorphisms is a rich area of active research. My contribution to this body work has been to show existence and uniqueness of a range of equilibrium measures, including measures of maximal entropy, for smooth pseudo-Anosov diffeomorphisms and almost-Anosov diffeomorphisms of the torus.

3.1 Pseudo-Anosov smooth models

In [2], M. Gerber and A. Katok provided an explicit procedure to construct a nonuniformly hyperbolic C^∞ diffeomorphism $g : M \to M$ on any Riemannian surface M. The construction is a perturbation of pseudo-Anosov homeomorphisms, which are locally uniformly hyperbolic surface homeomorphisms that are smooth except at finitely many singular fixed points. In the construction, the trajectories near these singularities are slowed down, so that the differentials become the identity on the tangent space. Unlike Anosov diffeomorphisms, which only exist on the torus in two dimensions, pseudo-Anosov diffeomorphisms may be constructed on any compact surface (as every compact surface admits pseudo-Anosov homeomorphisms).

In [15], I investigated the existence and uniqueness of equilibrium states for a family of observables for pseudo-Anosov diffeomorphisms. The observables I studied were the one-parameter family of *geometric t-potentials* defined by $\varphi_t(x) = -t \log \det \left|dg_x|_{E^u(x)}\right|$. In particular, the equilibrium states for φ_0 and φ_1 are measures of maximal entropy and SRB measures, respectively. I was able to prove the following result, addressing Problems 1 and 2 for a range of equilibrium states for pseudo-Anosov diffeomorphisms.

Theorem 1. *Let M be a compact Riemannian 2-manifold. Given any $t_0 < 0$, there is a pseudo-Anosov diffeomorphism $g : M \to M$ for which the following hold:*

1. *For any $t \in (t_0, 1)$, there is a unique equilibrium measure μ_t associated to φ_t. This equilibrium measure has exponential decay of correlations and satisfies the Central Limit Theorem with respect to a class of functions containing all Hölder continuous functions on M, and is Bernoulli. Additionally, the pressure function $t \mapsto P_g(\varphi_t)$ is real analytic in the open interval $(t_0, 1)$.*

2. *For $t = 1$, there are two classes of equilibrium measures associated to φ_1: convex combinations of Dirac measures concentrated at the singularities, and a unique SRB measure (which has density with respect to Riemannian volume).*

3. *For $t > 1$, the equilibrium measures associated to φ_t are precisely the convex combinations of Dirac measures concentrated at the singularities.*

My proof relies on the technology of Young towers, which are an adaptation of renewal theory to smooth ergodic theory. The thermodynamics and ergodic properties of these dynamical towers are an active area of research [10, 13], and are central to my work.

3.2 Almost-Anosov maps

Almost-Anosov diffeomorphisms were first described in [5], where H. Hu and L-S. Young investigated whether maps that are uniformly hyperbolic "essentially everywhere" admit SRB measures (addressing Problem 1). The answer to this question, as it turns out, is "not necessarily". Hu and Young constructed a diffeomorphism of the torus with uniform contraction along the stable leaves of the foliation, but with expansion that slowed down to 1 at a fixed point; in other words, the differential at the singular fixed point admitted eigenvalues of 1 and $\lambda < 1$. This "almost Anosov diffeomorphism" does not admit an SRB probability measure, but it does admit an "infinite SRB measure", or an invariant measure with positive Lyapunov exponents almost everywhere, absolutely continuous conditional measures on the foliation, and giving finite mass to $M \setminus U$ for every neighborhood U of the singular fixed point.

Then in [4], Hu gave the following definition of an almost-Anosov map:

Definition. A map $f \in \mathrm{Diff}^4(M)$ is *almost-Anosov* if there exist two families of cones $\mathcal{C}^u_x \subset T_xM$ and $\mathcal{C}^s_x \subset T_xM$ (with $x \mapsto \mathcal{C}^u_x$ and $x \mapsto \mathcal{C}^s_x$ continuous) such that, except at a finite set of fixed points,

- $df_x\mathcal{C}^u_x \subset \mathcal{C}^u_{f(x)}$ and $df_x\mathcal{C}^s_x \supset \mathcal{C}^s_{f(x)}$, and
- $|df_xv| > |v|$ for $v \in \mathcal{C}^u_x$, and $|df_xv| < |v|$ for $v \in \mathcal{C}^s_x$.

Hu considered almost-Anosov maps whose differential slowed down to the identity at a single fixed point, known as an *indifferent* fixed point. His work showed that even when both expansion and contraction slow down to zero, there are examples of almost-Anosov maps admitting an SRB probability measure, and examples admitting an infinite SRB measure.

In [14], using thermodynamics of Young towers, I proved a similar result to Theorem 1:

Theorem 2. *Given an almost-Anosov diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ of the torus with an indifferent fixed point at the origin, there is a $t_0 < 0$ so that for any $t \in (t_0, 1)$, there is a unique equilibrium measure μ_t associated to $\varphi_t(x) = -t \log \det \left| df|_{E^u(x)} \right|$. This equilibrium state has exponential decay of correlations and satisfies the central limit theorem with respect to a class of functions containing all Hölder continuous functions on $\mathbb{T}^2$. In particular, this range of equilibrium measures includes a unique measure of maximal entropy.*

4 Current work

4.1 Singular hyperbolic attractors

In addition to my work in thermodynamics of nonuniformly hyperbolic surface diffeomorphisms, I have also studied the ergodic properties of singular hyperbolic dynamical systems. These are dynamical systems whose underlying state space $K \subset M$ contains a set of singularities $N \subset K$ where the map $f : K \setminus N \to K$ fails to be continuous and/or fails to be differentiable. A standard example of such a map is the geometric Lorenz attractor on the square $K = (-1, 1)^2$, for which the function is discontinuous on the line $y = 0$ (specifically the continuous extensions of this function to the rectangles $[-1, 1] \times [-1, 0]$ and $[-1, 1] \times [0, 1]$ each collapse this line to a single point). The Lozi attractor is another such example, in which the function f is continuous on the singular set, but not differentiable.

In [8] it is shown that in general, singular hyperbolic maps admit SRB measures, but these measures may not be unique. One of my current projects addresses the following question:

Project 1. What are necessary and sufficient conditions to guarantee at most finitely many ergodic SRB measures? What are necessary and sufficient conditions to guarantee a unique SRB measure?

In [16], I used a Hopf argument to show that if the singular set N is a disjoint union of finitely many closed submanifolds with boundary, then with certain other regulatory conditions, the singular hyperbolic attractor admits at most finitely many SRB measures. The hypothesis that N consists of finitely many closed submanifolds is easy to verify for most examples of singular hyperbolic attractors. I was also able to provide conditions under which the SRB measure is unique.

Theorem 3. *Let Λ be a singular hyperbolic attractor of a map $f : K \setminus N \to K$. Suppose the singular set N is a disjoint union of finitely many closed submanifolds with boundary.*

(a) Under certain regularity conditions on the singular set N and the map f, the restricted map $f|_\Lambda : \Lambda \to \Lambda$ admits finitely many ergodic components, and Λ admits finitely many distinct ergodic SRB measures.

(b) If the stable foliation of f in K satisfies certain regulatory conditions, then the number of distinct ergodic SRB measures of f correspond to the components of topological transitivity in Λ. In particular, under these assumptions, $f|_\Lambda : \Lambda \to \Lambda$ is topologically transitive if and only if Λ admits exactly one ergodic SRB measure.

For non-transitive singular hyperbolic maps, the condition that N is a disjoint union of finitely many submanifolds is necessary in general. For example, one can consider the classic geometric Lorenz system $f : (-1, 1)^2 \setminus ((-1, 1) \times \{0\}) \to (-1, 1)^2$, and embed countably many copies of this system into subsets $(-1, 1)^2$ of the form $K_n = (-1, 1) \times \left(2^{-n} - 1, 2^{-(n-1)} - 1\right)$, $n \geq 0$. Each copy admits is own SRB measure. However, this example is not topologically transitive and not particularly informative. Within this current project, one of my goals is constructing a transitive example of a singular hyperbolic attractor whose singular set has infinitely many components.

4.2 Further statistical properties of pseudo-Anosov maps

The results in [10] on Young towers, which I used to prove exponential decay of correlations in Theorems 1 and 2 for measures of maximal entropy, do not apply to the SRB measure for pseudo- and almost-Anosov diffeomorphisms. This is one direction in which I am currently trying to develop this work:

Project 2. Given a pseudo-Anosov diffeomorphism $g : M \to M$ of a compact Riemannian 2-manifold M, the g-invariant Riemannian area on M is an SRB measure. What is the decay rate of the correlations with respect to this area?

In [9], Y. Pesin, S. Senti, and F. Shahidi showed that the Lebesgue measure μ_1 for the Katok map of the torus has polynomial decay of correlations. The Katok map is an early example of a nonuniformly hyperbolic surface diffeomorphism, originally described in [7], and its construction is similar to the construction of the pseudo-Anosov diffeomorphisms described in [2]. For this reason, it seems likely that pseudo-Anosov diffeomorphisms also admit polynomial decay of correlations with respect to their invariant Riemannian area.

Additionally, my current results on pseudo-Anosov maps only yield a range of equilibrium states up to a fixed $t_0 < 0$. In [17], it is shown that the Katok map $G : \mathbb{T}^2 \to \mathbb{T}^2$ admits a unique equilibrium state for $\varphi_t(x) = -t \log \det \left| dG|_{E^u(x)} \right|$ for *every* $t < 0$.

Project 3. Let $g : M \to M$ be a pseudo-Anosov diffeomorphism of a Riemannian manifold M. Does there exist a unique equilibrium state for $\varphi_t(x) = -t \log \det \left| dg|_{E^u(x)} \right|$?

The exact techniques in [9] cannot be applied directly to the pseudo-Anosov case; the slow-down procedure for the Katok map is generally speaking a more significant perturbation of a hyperbolic toral automorphism than the perturbation of a pseudo-Anosov homeomorphism to produce g. Therefore the technical arguments are notably different, and more investigation is required in order to study the statistical properties of this measure.

5 Future work: Other low-dimensional nonuniform hyperbolicity

I have also considered the thermodynamics of maps admitting both elliptic and hyperbolic behavior, such as the "elliptic island" systems described in [3] and [12].

Project 4. Under what conditions do maps admitting both elliptic and hyperbolic behavior also admit Young towers? What are the statistical properties of these maps with respect to SRB measures, measures of maximal entropy, and other equilibrium states?

Dynamical systems exhibiting both elliptic and hyperbolic behavior are famously difficult to analyze. A well-known example is the *Chirikov Standard Map* $f_k : \mathbb{T}^2 \to \mathbb{T}^2$, defined by

$$f_k(x, y) = (2x - y + k \sin(x), x).$$

Part of what makes analyzing the Standard Map and other "elliptic island" maps difficult is that unlike pseudo-Anosov diffeomorphisms, the angle between the subspaces $E^s(x)$ and $E^u(x)$ may become arbitrarily small, potentially obstructing nonuniform hyperbolicity. It is an open problem, for instance, to show that the Standard Map is nonuniformly hyperbolic for a Lebesgue-positive set of parameters k.

However, there are examples of toral maps that exhibit both elliptic and hyperbolic behavior, and are known to be nonuniformly hyperbolic. It is possible that the techniques used in [15] and [14] may be used to show that such maps admit SRB measures, measures of maximal entropy, and other equilibrium measures. This may provide insight into other ways of analyzing the Standard Map and other "elliptic island" systems.

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