SRB MEASURES OF SINGULAR HYPERBOLIC ATTRACTORS.

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Abstract. It is known that nonuniformly hyperbolic maps admitting singularities have at most countably many ergodic Sinai-Ruelle-Bowen (SRB) measures. These maps include the Belykh attractor, the geometric Lorenz attractor, and more general Lorenz-type systems. In this paper, we establish easily verifiable sufficient conditions guaranteeing that the number of ergodic SRB measures is at most finite, and provide examples and nonexamples showing that the conditions are necessary in general.

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1. Introduction

One primary question in smooth ergodic theory is the existence of "physical measures" for a smooth dynamical system. Given a compact Riemannian manifold M and a smooth map $f : U \to M$, $U \subseteq M$ open, a *physical measure* is one in which the Birkhoff averages of continuous functions are constant on a set of positive measure. In other words, a probability measure μ is a *physical measure* if

$$m\left\{x \in U : \lim_{n\to+\infty} \frac{1}{n}\sum_{k=0}^{n-1} \left(\varphi \circ f^k\right)(x) = \int_U \varphi \, d\mu \quad \forall \varphi \in C^0(U)\right\} > 0,$$

where m is the Riemannian volume. Among the most significant physical measures are the *Sinai-Ruelle-Bowen (SRB) measures.* These are invariant measures for hyperbolic dynamical systems that (a) admit positive Lyapunov exponents almost everywhere, and (b) have conditional measures on unstable leaves that are absolutely continuous with respect to the Riemannian leaf volume. For uniformly hyperbolic dynamical systems (such as Anosov and Axiom A diffeomorphisms), there is a unique SRB measure [7], and the existence of SRB measures has been established for several classes of nonuniformly hyperbolic dynamical systems [8, 12, 16]. It was further shown in [13] that if M is a compact Riemannian 2-manifold and $f : M \to M$ is a hyperbolic diffeomorphism admitting an SRB measure, then this SRB measure is unique.

Many dynamical systems in engineering and natural sciences exhibit "chaotic" behavior: their trajectories appear disordered and they are highly sensitive to initial data. The simplest mathematical examples of such systems are uniformly hyperbolic and uniformly expanding, and so hyperbolic dynamical systems have been at the forefront of smooth ergodic theory since at least the 1960s. Unfortunately, most stochastic dynamical systems arising from physical and natural phenomena are not uniformly hyperbolic. In these instances, uniqueness results for uniformly hyperbolic dynamical systems and surface maps (such as those described in [13]) no longer apply. Examples of such dynamical systems include the Lorenz attractor model of atmospheric convection and the Belykh attractor of phase synchronization theory [4, 11, 15]. These are maps of the unit square that admit highly complex limit sets, so that the resulting maps are not invariant under Lebesgue volume and may not *a priori* admit a unique SRB measure.

Although our results concern discrete singular hyperbolic maps, historically many results about singular hyperbolic attractors come from investigations into hyperbolic flows, the most famous being the flow generated by the Lorenz equations. In [9], J. Kaplan and J. Yorke used a Poincaré return map to study the dynamical behavior of the Lorenz attractor, such as the parameters for which periodic points are dense. This Poincaré map was later reformulated as the *geometric Lorenz attractor*, which is a simplified discrete model of the Poincaré map of the original Lorenz flow. The more general family of discrete *Lorenz-type maps* was introduced in [1]. In the years that followed, the Lorenz system and related hyperbolic flows have led to active research in singular hyperbolic attractors (see e.g. [1, 4, 11, 14, 15], and others). There is also a large body of work on singular hyperbolic and sectional-hyperbolic flows more generally. In [15], it is shown that singular hyperbolic flows admit finitely many ergodic physical measures; more recently, it was shown in [2] that flows of Hölder-C^1 vector fields admitting a sectional-hyperbolic attracting set admit finitely many ergodic SRB measures. The proof in [2] also relies on Poincaré return maps, and so these results extend to discrete singular hyperbolic maps arising as Poincaré maps of hyperbolic flows. For a detailed discussion of the ergodic properties of hyperbolic flows and their attractors, see [3].

In this paper, we consider the class of discrete singular hyperbolic dynamical systems. These are hyperbolic maps $f : K \setminus N \to K$, where $K \subset M$ is a precompact open subset of a Riemannian manifold M, and $N \subset K$ is a closed subset of singularities on which f fails to be continuous and/or differentiable. The map f is uniformly hyperbolic on the non-invariant set $K \setminus N$, but behaves more similarly to the non-uniformly hyperbolic setting on an invariant set that consists of trajectories passing nearby the the singular set N with a prescribed rate. Our setting includes systems that are derived from Poincaré maps of hyperbolic flows, such as the geometric Lorenz attractor, but also includes singular hyperbolic dynamical systems that do not arise from flows, such as the Lozi map [10]. In [11], it was shown that the attractors admitted by singular hyperbolic maps support at most countably many ergodic SRB measures. In [14], conditions were given under which a singular hyperbolic attractor admits at most finitely many ergodic SRB measures. We provide an alternative proof of this result, with somewhat different conditions that are easy to verify. Namely, in addition to a set of standard regulatory conditions (see conditions (H1) - (H4), also in [11]), if the singular set is a disjoint union of finitely many embedded submanifolds that transversally intersect unstable leaves, and if the image of neighborhoods of the singular set remain separated from the singular set under the dynamics for sufficiently long but finite time (conditions (H5), (H8), and (H9)), then there are at most finitely SRB measures.

Although conditions (H5) - (H9) are fairly common in the literature on singular hyperbolic attractors (see the examples in [11]), there exist singular hyperbolic attractors that do not satisfy these conditions and admit infinitely many ergodic components. For this reason, these conditions are necessary for our statement to be true in full generality. For example, one can construct a family of Lorenz-type attractors whose singular sets have infinitely many components, and these examples admit countably many SRB measures, but these maps are not topologically transitive (see Section 4.1). We are at this time unaware of an example of a singular hyperbolic map that is topologically transitive on its attractor, and admits countably infinitely many ergodic SRB measures, or even more than one ergodic SRB measure.

This paper is structured as follows. Section 2 is devoted to preliminary constructions and definitions needed to discuss singular hyperbolic dynamical systems. Our main result is stated and proven in Section 3. Section 4 is spent discussing examples of dynamical systems satisfying the hypotheses of our main result, as well as examples of systems that fail these hypotheses and that admit infinitey many SRB measures.

2. Preliminaries

We begin by defining singular hyperbolic attractors, and discuss some of their major properties. We consider a Riemannian manifold M, an open, bounded, connected subset $K \subset M$ with compact closure, and a closed subset $N \subset K$. We further consider a map $f : K \setminus N \to K$ satisfying:

(H1) f is a C^2 diffeomorphism from $K \setminus N$ to $f(K \setminus N)$.

We further define $N^+ := N \cup \partial K$ as the *discontinuity set* for f (on which the function f is discontinuous), and further define

$$N^- = \left\{ y \in K : \text{There are } z \in N^+ \text{ and } z_n \in K \setminus N^+ \text{ s.t. } z_n \to z \text{ and } f(z_n) \to y \right\}.$$

The set N^- is referred to as the *discontinuity set* for f^{-1}. We further assume the map f satisfies:

(H2) There exist $C_i > 0$ and $\alpha_i \geq 0$, with $i = 1, 2$, such that

$$\left\| d^2 f_x \right\| \leq C_1 \rho(x, N^+)^{-\alpha_1} \quad \text{for } x \in K \setminus N,$$
$$\left\| d^2 f_x^{-1} \right\| \leq C_2 \rho(x, N^-)^{-\alpha_2} \quad \text{for } x \in f(K \setminus N)$$

where ρ is the Riemannian distance in M.

Define the set K^+ by

$$K^+ = \bigcap_{n=0}^{\infty} \left(K \setminus f^{-n}(N^+) \right) = \left\{ x \in K : f^n(x) \notin N^+ \text{ for all } n \geq 0 \right\},$$

so that K^+ is the largest open forward-invariant set on which f is continuous. Further, define

$$D = \bigcap_{n=0}^{\infty} f^n(K^+) \quad \text{and} \quad \Lambda = \overline{D}.$$

We say Λ is the *attractor* for f.

Proposition 2.1. [11] *We have $D = \Lambda \setminus \bigcup_{n \in \mathbb{Z}} f^n(N^+)$. Furthermore, f and f^{-1} are well-defined on D, and $f(D) = D$ and $f^{-1}(D) = D$.*

Given $z \in M$, $\alpha > 0$, and a subspace $P \subset T_z M$, we denote the cone at z around P with angle α by

$$C(z, \alpha, P) = \left\{ v \in T_z M : \angle(v, P) := \inf_{w \in P} \angle(v, w) \leq \alpha \right\}.$$

Definition 2.2. The set Λ defined above is a *singular hyperbolic attractor* if there is $C > 0$, $\lambda > 1$, a function $\alpha : D \to \mathbb{R}$, and two distributions P^s, P^u on $K \setminus N^+$ of dimensions $\dim P^s = p$, $\dim P^u = q = n - p$ (with $n = \dim M$), such that the cones $C^s(z) = C(z, \alpha(z), P^s_z)$ and $C^u(z) = C(z, \alpha(z), P^u_z)$ satisfy the following conditions:

(a) The angle between $C^s(z)$ and $C^u(z)$ is uniformly bounded below over $K \setminus N^+$, and in particular, $C^s(z) \cap C^u(z) = 0$;

(b) $df_z(C^u(z)) \subset C^u(f(z))$ for $z \in K \setminus N^+$, and $df_z^{-1}(C^s(z)) \subset C^s(f^{-1}(z))$ for $z \in f(K \setminus N^+)$;

(c) for any $n > 0$, we have:

$$|df_z^n v| \geq C\lambda^n |v| \quad \text{for } z \in K^+, v \in C^u(z);$$
$$|df_z^{-n}| \geq C\lambda^n |v| \quad \text{for } z \in f^n(K^+), v \in C^s(z).$$

Define the following subsets of T_zM for $z \in D$:

$$E^s_z = \bigcap_{n=0}^{\infty} df^{-n}_{f^n(z)} C^s(f^n(z)) \quad \text{and} \quad E^u_z = \bigcap_{n=0}^{\infty} df^n_{f^{-n}(z))} C^u(f^{-n}(z)).$$

Proposition 2.3. [11] *The sets E^s_z and E^u_z are subspaces of T_zM, called the* stable *and* unstable subspaces *at z respectively. They satisfy the following properties:*

(a) the dimensions of these subspaces are the same as the respective subspaces P^s_z and P^u_z around which the cones $C^s(z)$ and $C^u(z)$ are centered. That is, $\dim E^s_z = \dim P^s_z = p$ and $\dim E^u_z = \dim P^u_z = q = n - p$;

(b) $T_zM = E^s_z \oplus E^u_z$;

(c) the angle between E^s_z and E^u_z is bounded below uniformly over D;

(d) for any $n \geq 0$ and $z \in D$, we have

$$|df_z^n v| \leq C\lambda^{-n}|v| \quad \text{for } v \in E^s(z),$$
$$|df_z^{-n} v| \leq C\lambda^{-n}|v| \quad \text{for } v \in E^u(z).$$

The distributions E^s and E^u on D thus form uniformly hyperbolic structure with singularities. In particular, they are the tangent spaces of stable and unstable foliations on D. To rigorously characterize the leaves of these foliations, we need to define the subsets on which stable and unstable manifolds may be defined.

For arbitrary $\varepsilon > 0$ and $l \in \mathbb{N}$, we denote:

$$\widehat{D}^+_{\varepsilon,l} = \left\{z \in K^+ : \rho\left(f^n(z), N^+\right) \geq l^{-1}e^{-\varepsilon n},\ n \geq 0\right\};$$
$$D^-_{\varepsilon,l} = \left\{z \in \Lambda : \rho\left(f^{-n}(z), N^-\right) \geq l^{-1}e^{-\varepsilon n}, n \geq 0\right\};$$
$$D^+_{\varepsilon,l} = \widehat{D}^+_{\varepsilon,l} \cap \Lambda;$$
$$D^0_{\varepsilon,l} = D^-_{\varepsilon,l} \cap D^+_{\varepsilon,l};$$
$$D^\pm_\varepsilon = \bigcup_{l\geq 1} D^\pm_{\varepsilon,l};$$
$$D^0_\varepsilon = \bigcup_{l\geq 1} D^0_{\varepsilon,l}.$$

We note that $\widehat{D}^+_{\varepsilon,l}$, $D^\pm_{\varepsilon,l}$, and $D^0_{\varepsilon,l}$ are closed, and hence compact. Also observe that $D^0_\varepsilon = D^+_\varepsilon \cap D^-_\varepsilon \subset D$ for $\varepsilon > 0$, and D^0_ε is invariant under both f and f^{-1}. Further, D^+_ε and D^-_ε are invariant under f and under f^{-1} respectively.

Definition 2.4. The attractor Λ is *regular* if the following condition holds:

(H3) $D^0_\varepsilon \neq \emptyset$ for sufficiently small $\varepsilon > 0$.

Now define the functions $\varphi(z) = \rho(z, N^+)$ and

$$\chi^{\pm}_{\varphi} = \limsup_{n\to\pm\infty} \log\varphi\left(f^n(z)\right).$$

Then $\chi^{\pm}_{\varphi}$ are invariant under both f and f^{-1} on D, and $\chi^{\pm}_{\varphi}(z) \le 0$ on D. Furthermore, by definition of $\chi^{\pm}_{\varphi}$, for any $\varepsilon > 0$ and $z \in D$, there exists $K = K(\varepsilon, z) > 0$ such that for any $n \in \mathbb{Z}$,

$$\rho\left(f^n(z), N\right) = \varphi\left(f^n(z)\right) \le K e^{\left(\chi^{\pm}_{\varphi}(z)+\varepsilon\right)|n|}.$$

Define the sets

$$\widetilde{D}^{\pm} = \left\{ z \in D : \chi^{\pm}_{\varphi}(z) = \lim_{n\to\pm\infty} \frac{1}{|n|}\log\varphi\left(f^n(z)\right) = 0 \right\},$$
$$\widetilde{D}^0 = \widetilde{D}^+ \cap \widetilde{D}^-.$$

These sets are invariant under both f and f^{-1}. Furthermore, for any $\varepsilon > 0$ and $z \in \widetilde{D}^+$ (respectively $z \in \widetilde{D}^-$ or $z \in \widetilde{D}^0$), there exists $C = C(\varepsilon, z)$ such that for $n > 0$ (respectively $n < 0$ or $n \in \mathbb{Z}$),

$$\rho\left(f^n(z), N\right) = \varphi\left(f^n(z)\right) \ge C e^{-\varepsilon|n|}.$$

Denote by $\mathcal{M}_f$ the set of Borel f-invariant probability measures on Λ.

Proposition 2.5. [11] *If there exists $\mu \in \mathcal{M}_f$ such that $\mu(D) > 0$ and*

$$\left|\int_\Lambda \log\varphi(z)\, d\mu(z)\right| < \infty,$$

then $\mu\big(\widetilde{D}^0\big) > 0$, and in particular, Λ is regular.

Denote by $U(\varepsilon, N^+)$ the ε-neighborhood in K of N^+, and by ν the Riemannian volume in K.

Proposition 2.6. [11] *Assume that there is $C > 0$ and $q > 0$ such that for any $\varepsilon > 0$ and $n > 0$, we have*

$$\nu\left(f^{-n}(U(\varepsilon, N^+)) \cap f^n(K^+)\right) \le C\varepsilon^q.$$

Then Λ is regular.

We are now ready to define the stable and unstable submanifolds on D. For the proof of the following proposition, see the discussion in Sections 1.5 and 2.1 of [11].

Proposition 2.7. *There exists $\varepsilon > 0$ and such that:*

(a) For $z \in D^+_\varepsilon$, there is an embedded (possibly disconnected) submanifold $W^s_{\text{loc}}(z)$ of dimension $p = \dim E^s_z$ for which $T_z W^s(z) = E^s_z$;

(b) For $z \in D^-_\varepsilon$, there is an embedded (possibly disconnected) submanifold $W^u_{\text{loc}}(z)$ of dimension $q = n - p = \dim E^u_z$ for which $T_z W^u(z) = E^u_z$.

Furthermore, define $B^s_z(y, r)$ to be the ball in $W^s(z)$ of radius r centered at $y \in W^s_{\text{loc}}(z)$, where the distance is the induced distance ρ^s on $W^s_{\text{loc}}(z)$. Define $B^u_z(y, r)$ and ρ^u similarly. Then there is an α with $\lambda^{-1} < \alpha < 1$ such that for $r > 0$, there is a constant $C = C(r)$ such that:

(c) for $z \in D^+_\varepsilon$, $y \in W^s_{\text{loc}}(z)$, $w \in B^s_z(y, r)$, and $n \ge 0$, we have

$$\rho^s(f^n(y), f^n(w)) \le C\alpha^n \rho^s(y, w);$$

(d) for $z \in D^-_\varepsilon$, $y \in W^u_{\mathrm{loc}}(z)$, $w \in B^u_z(y,r)$, and $n \le 0$, we have
$$\rho^u(f^n(y), f^n(w)) \le C\alpha^n \rho^u(y,w).$$

Additionally, for $z \in D^-_{\varepsilon,l}$, let $B(z,\delta)$ denote the ball of ρ-radius δ centered at z. Then there are $\delta_i = \delta_i(z) > 0$, $i = 1,2,3$, with $\delta_1 > \delta_2 > \delta_3$, so that for $w \in B(z,\delta_3)$, the intersection $B^s_z(z,\delta_1) \cap W^u_{\mathrm{loc}}(w)$ is nonempty and contains exactly one point, denoted $[w,z]$; and furthermore, $B^u_w([w,z],\delta_2) \subset W^u_{\mathrm{loc}}(w)$.

We denote
$$W^u(x) = \bigcup_{n\ge 0} f^n\left(W^u_{\mathrm{loc}}(f^{-n}(x))\right)$$
for $x \in K$ and
$$W^s(x) = \bigcup_{n\ge 0} f^{-n}\left(W^s_{\mathrm{loc}}(f^n(x)) \cap \Lambda\right)$$
for $x \in \Lambda$.

Given $\delta > 0$ and $x \in K$, let $B^u_T(\delta,x) \subset E^u_x$ denote the open ball of radius δ in E^u_x. For δ less than the injectivity radius of M at x, suppose the connected component of $\left(\exp_x|_{B^u_T(\delta,x)}\right)^{-1}(W^u(x)) \subset T_xM$ containing 0 is the graph of some smooth function $\psi : B^u_T(\delta,x) \to E^s_x$. If such a ψ exists for a particular $x \in M$ and $\delta > 0$, we denote
$$W^u_\delta(x) = \exp_x\left(\{(u,\psi(u)) : u \in B^u_T(\delta,x)\}\right).$$
Such a number $\delta > 0$ and such a function ψ exist for each particular $x \in K$ (in particular they form $W^u_{\mathrm{loc}}(x)$), but δ may depend on x (and in particular may not have a uniform lower bound). We define $W^s_\delta(x)$ similarly.

Given local submanifolds $W^s_{\mathrm{loc}}(z_1)$ and $W^s_{\mathrm{loc}}(z_2)$, we define the *holonomy map* $\pi : W^s_{\mathrm{loc}}(z_1) \to W^s_{\mathrm{loc}}(z_2)$ to be $\pi(w) = [w,z_2] = W^u_{\mathrm{loc}}(w) \cap W^s_{\mathrm{loc}}(z_2)$. Let $\nu^s_z = \nu|_{W^s_{\mathrm{loc}}(z)}$ and $\nu^u_z = \nu|_{W^u_{\mathrm{loc}}(z)}$ denote the induced Riemannian volumes on $W^s_{\mathrm{loc}}(z)$ and $W^u_{\mathrm{loc}}(z)$ respectively for $z \in D^\pm_\varepsilon$.

Proposition 2.8. *The local foliation $W^s_{\mathrm{loc}}(z)$ for $z \in D^0_{\varepsilon,l}$ is absolutely continuous, in the sense that for any $z_1, z_2 \in D^0_{\varepsilon,l}$, the pushforward measure $\pi_*\nu^s_{z_1}$ on $W^s_{\mathrm{loc}}(z_2)$ is absolutely continuous with respect to $\nu^s_{z_2}$.*

Proof. This follows from Proposition 10 of [11]. □

We make the following assumption on the attractor:

(H4) There are numbers $\beta > 0$, $\delta > 0$, $\varepsilon > 0$, $r_0 > 0$, $C > 0$, and $t > 0$ and a point $z \in D^-_\varepsilon$ such that for any $r \in (0, r_0]$, and any $n \ge 0$,
$$\nu^u_z\left(W^u_{\mathrm{loc}}(z) \cap f^{-n}(B_r(N^+))\right) \le C\beta^t.$$

Generally, maps satisfying (H1) and (H2) are dissipative, and so do not preserve Riemannian volume. Therefore, our interest is in the following class of measures on K:

Definition 2.9. A probability measure μ on K is an *SRB (Sinai-Ruelle-Bowen) measure* if μ is f-invariant and if the conditional measures on the unstable leaves are absolutely continuous with respect to the Riemannian leaf volume.

The following result comes from Theorems 1 and 2 of [11]:

Proposition 2.10. *Let Λ be a singular hyperbolic attractor satisfying Conditions (H3) and (H4). Then there is an SRB measure μ concentrated on D satisfying the hypotheses of 2.5. Furthermore, Λ decomposes into countably many ergodic components $\Lambda_0, \Lambda_1, \ldots$, each of which may be further partitioned into finitely many subsets $\Lambda_i^1, \ldots, \Lambda_i^{k_i}$ with $f\left(\Lambda_i^j\right) = \Lambda_i^{j+1}$, $f\left(\Lambda_i^{k_i}\right) = \Lambda_i^1$, and $f^{k_i}|_{\Lambda_i^1}$ is isomorphic to a Bernoulli automorphism.*

Remark 2.11. Each of the ergodic components in the preceding proposition is f-invariant, and so admits a unique ergodic SRB measure μ_i. Thus every (not necessarily ergodic) SRB measure is a (possibly infinite) convex combination of the measures μ_i.

We will describe the construction of the SRB measures in Proposition 2.10. Let $J^u(z) = \det\left(df|_{E_z^u}\right)$ denote the *unstable Jacobian* of f at a point $z \in D$. For $y \in W^u(z)$ and $n \geq 1$, set

$$\kappa_n(z, y) = \prod_{j=0}^{n-1} \frac{J^u\left(f^{-j}(z)\right)}{J^u\left(f^{-j}(y)\right)}.$$

The functions κ_n converge pointwise to a function κ (see [11], Proposition 6(1)). Fix $z \in D_\varepsilon^-$ and a sufficiently small $r > 0$, and set

$$U_0 := B^u(z, r) := B_z^u(z, r), \quad \widetilde{U}_n := f(U_{n-1}), \quad U_n := \widetilde{U}_n \setminus N^+.$$

Further set

$$\widetilde{C}_0 = 1 \quad \text{and} \quad \widetilde{C}_n = \left(\prod_{k=0}^{n-1} J^u\left(f^k(z)\right)\right)^{-1}.$$

For $n \geq 0$, define the measures $\widetilde{\nu}_n$ on U_n by

$$d\widetilde{\nu}_n(y) = \widetilde{C}_n \kappa\left(f^n(z), y\right) d\nu_z^u(y),$$

and let ν_n be a measure on Λ defined by $\nu_n(A) = \widetilde{\nu}_n(A \cap U_n)$ for any Borel $A \subseteq \Lambda$. Under moderate assumptions, we have that $\nu_n = f_*^n \nu_0$ (see [11], Proposition 8). Consider the sequence of measures

$$\mu_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu_k.$$

These measures admit a subsequence that converges in the weak topology to an f-invariant SRB measure on Λ, proving existence of SRB measures.

3. Main Result

We begin by reviewing the assumptions we make on our map $f : K \setminus N \to K$. Assumption (H1) is the basic setting, and assumptions (H2) - (H4) concern the regularity of the map. We now complete the assumptions of our setting. Below, assumption (H5) concerns the structure of the singular set N; assumptions (H6) - (H8) concern the smoothness and hyperbolicity of f; and assumption (H9) is a further regulatory assumption. These assumptions correspond to Assumptions (H1) - (H9) in [11].

(H5) The singular set N is the disjoint union of finitely many embedded submanifolds N_i with boundary, of dimension equal to the codimension of the unstable foliation W^u.

(H6) f is continuous and differentiable in $K \setminus N^+$.
(H7) f possesses two families of stable and unstable cones $C^s(z)$, $C^u(z)$, for $z \in K \setminus N^+$.
(H8) The assignment $z \mapsto C^u(z)$ has a continuous extension in each $\overline{K}_i \subset K$ (where K_i are the connected components of $K \setminus N^+$), and there exists $\alpha > 0$ such that for $z \in N \setminus \partial K$ and $v \in C^u(z)$, $w \in T_z N$, we have $\angle(v, w) \geq \alpha$.
(H9) $f^j(N^-) \cap N^+ = \emptyset$ for $0 \leq j < k$, where $\lambda^k > 2$ and

$$\lambda = \inf_{x \in K \setminus N^+} \|df_x\| > 1.$$

We now state our main result.

Theorem 3.1. *Let Λ be a singular hyperbolic attractor of a map $f : K \setminus N \to K$ satisfying conditions (H1) and (H5) - (H9). Then $f : \Lambda \to \Lambda$ admits at most finitely many ergodic components, and in particular, Λ admits finitely many distinct ergodic SRB measures.*

We have defined $W^u_\delta(x)$ to be the image under $\exp_x : T_x M \to M$ of the graph of a function $\psi : B^u_T(\delta, x) \to E^s_x$, where $B^u_T(\delta, x) \subset E^u_x$ is the open ball of radius δ centered at $0 \in E^u_x$, provided that such a function ψ and such a number $\delta > 0$ exist. For each $x \in M$, such a ψ and δ do exist. However, we may also fix $\delta > 0$ and define the set B^-_δ to be the set of all $x \in D$ for which there is some $\varepsilon > 0$, some $l \in \mathbb{N}$, and some $y \in D^-_{\varepsilon,l}$ so that $W^u_\delta(y)$ exists and contains x. Note that $x \notin B^-_\delta$ if, for example, $W^u(x)$ is not the image under $\exp_x$ of a smooth graph in a δ-neighborhood of $0 \in T_x M$.

Lemma 3.2. *For sufficiently small $\delta > 0$, the set B^-_δ admits at most finitely many ergodic SRB measures.*

Proof. The proof is an adaptation of a Hopf argument. Given a continuous function $\varphi \in C^0(K)$, let $\varphi_+ : K \to \mathbb{R}$ and $\varphi_- : \Lambda \to \mathbb{R}$ denote the Birkhoff averages

$$\varphi_+(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi\left(f^k(x)\right) \quad \text{and} \quad \varphi_-(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi|_\Lambda\left(f^{-k}(x)\right).$$

assuming these limits exist. Let $\Lambda^+ \subseteq K$ denote the set of points where φ_+ exists for every $\varphi \in C^0(K)$, and let $\Lambda^- \subseteq \Lambda$ denote the set of points where φ_- exists for every $\varphi \in C^0(K)$. By the Birkhoff ergodic theorem, both Λ^+ and Λ^- have full measure with respect to any invariant measure. Observe that if $x \in \Lambda^-$ and $y \in W^u_\alpha(x)$ for $\alpha > 0$, then $\varphi_-(y) = \varphi_-(x)$, so $y \in \Lambda^-$, and so $W^u_\delta(x) \subseteq \Lambda^-$ for every $x \in \Lambda^-$. Similarly, $W^s_\alpha(x) \subseteq \Lambda^+$ for $x \in \Lambda^+$, $\alpha > 0$.

Recall that a point $x \in K$ lies in B^-_δ if and only if there is a $y = y(x) \in D^-_{\varepsilon,l}$ for some ε, l for which $x \in W^u_\delta(y)$. So let Λ^0 be the set of points $x \in B^-_\delta$ for which there is a subset $A \subseteq W^u_\delta(y)$ of full conditional measure (with respect to Lebesgue) such that $A \subseteq \Lambda^+$ and $\varphi_+|_A$ is constant for all $\varphi \in C^0(K)$.

We make the following claims:

- the set Λ^0 has full measure with respect to any invariant measure, and
- the set Λ^0 is closed.

Granting these claims for now, using the notation in the above paragraph, for $x \in \Lambda^0$, let $\varphi_+(W^u_{\text{loc}}(x)) = \varphi_+(z)$ for $z \in A \subseteq W^u_\delta(y)$, where $y = y(x)$ is as in the definition of B_δ^-. We will say that $x, z \in \Lambda^0$ are *equivalent*, and write $x \sim z$, if $\varphi_+(W^u_{\text{loc}}(x)) = \varphi_+(W^u_{\text{loc}}(z))$.

Note that the stable foliation W^s is absolutely continuous. So if $x \in \Lambda^0$ and $z \in W^s(x) \cap \Lambda^0$, then $\varphi_+(x) = \varphi_+(z)$, and there is a set A' of full measure in $W^u_\delta(y(z))$ on which φ_+ is constant and equal to $\varphi_+(W^u_{\text{loc}}(x))$. So $x \sim z$ whenever $z \in W^s(x) \cap \Lambda^0$.

Suppose Λ^0_1 is an equivalence class, and let $x \in \Lambda^0_1$. We claim there is an $\varepsilon > 0$ so that if $y \in \Lambda^0$ lies in the ε-ball centered at x, then $x \sim y$.

Indeed, by virtue of Proposition 7 in [11], there is an $\varepsilon > 0$ for which $B_\varepsilon(x)$ has *local hyperbolic product structure:* for $z \in W^u_\varepsilon(x)$ and $y \in W^s_\varepsilon(x)$, the intersection $W^u_\varepsilon(y) \cap W^s_\varepsilon(z)$ contains exactly one point, which we denote $[y, z]$. Let $y \in B_\varepsilon(x) \cap \Lambda^0$, let $B^u_\varepsilon(x)$ denote the ball in the local unstable manifold $W^u_{\text{loc}}(x)$ centered at x of size ε, and let $\theta : B^u_\varepsilon(x) \to W^u_\varepsilon(y)$ denote the holonomy map $\theta(z) = [y, z]$.

To show $x \sim y$, let $A \subseteq B^u_\varepsilon(x)$ be the set of points z for which $\varphi_+(z) = \varphi_+(x)$ for every continuous function φ. By definition of the set Λ^0, the set A has full measure in $B^u_\varepsilon(x)$. By absolute continuity of the stable foliation, $\theta(A)$ has full measure in $\theta(B^u_\varepsilon(x)) \subset W^u_\varepsilon(y)$. Since φ_+ is constant on stable leaves, $\varphi_+ \circ \theta = \varphi_+$ for every continuous φ. Therefore $\varphi_+(z_1) = \varphi_+(x)$ for almost every $z_1 \in \theta(B^u_\varepsilon(x))$. Again, by definition of Λ^0, $\varphi_+(z_1) = \varphi_+(y)$ for every continuous φ and almost every $z_1 \in \theta(B^u_\varepsilon(x))$. Therefore $\varphi_+(x) = \varphi)_+(y)$, and so $x \sim y$.

It follows from these arguments that each equivalence class is open in Λ^0, and hence is also closed in Λ^0. By closedness of Λ^0 in K, there is an $\varepsilon > 0$ such that each pair of equivalence classes is separated by a distance of at least ε. By compactness of K, it follows that there may only be finitely many such equivalence classes. Because an ergodic SRB measure must be supported on exactly one of these equivalence classes, there may only be finitely many SRB measures.

It remains only to prove our previous two claims. Let $\widehat{\Lambda}$ denote those points $x \in \Lambda^-$ such that $\varphi_-(x) = \varphi_+(x)$ for every continuous function φ. By the Birkhoff ergodic theorem, $\widehat{\Lambda}$ has full measure with respect to any invariant measure. Further, let $\widehat{\Lambda}^0$ denote the set of points $x \in \widehat{\Lambda}$ such that there is a set $A \subset W^u_\gamma(x)$ of full conditional measure such that $A \subseteq \Lambda^+$ and $\varphi_-(z) = \varphi_+(z)$ for every continuous function φ and all points $z \in A$. Here, $W^u_\gamma(x)$ is the connected component of $W^u(x)$ intersecting $\widehat{\Lambda}$ and containing x (and $W^s_\gamma(x)$ is defined similarly). Since $\varphi_-(z)$ takes the same value for all $z \in W^u_\varepsilon(x)$, $\varphi_-|_A = \varphi_+|_A$ is constant.

For $x \in \widehat{\Lambda}^0$, the set $\widehat{\Lambda}^0$ contains the union over $y \in W^s_\gamma(x) \cap B^-_\delta$ of manifolds $W^u_\varepsilon(y)$ that contain a subset of full conditional measure (as this subset lies in $\widehat{\Lambda}$, which has full measure). Because $W^s_\gamma(x)$ has full conditional measure in $\widehat{\Lambda}$, it follows that this union also has full measure, from which it follows that $\widehat{\Lambda}^0$ has full measure.

If a point x has a negative semitrajectory that enters $\widehat{\Lambda}^0$ infinitely often, then one can show $x \in \Lambda^0$. Since $\widehat{\Lambda}^0$ has full measure, the set of points whose negative semitrajectories enter $\widehat{\Lambda}^0$ also has full measure, so Λ^0 has full measure. This proves the first of the two previous claims.

To show that Λ^0 is closed, suppose x is the limit point of a sequence (x_n) in Λ^0. Since the stable foliation is absolutely continuous and $W^u_\varepsilon(x_n)$ converges to $W^u_\varepsilon(x)$ for $\varepsilon > 0$ small, we can find a set $A \subset W^u_\delta(y)$ of full conditional measure, where $y = y(x)$ is as in the definition of B^-_δ, such on which φ_+ is constant for all continuous functions φ. Therefore $x \in \Lambda^0$. □

Observe that if $\delta_1 < \delta_2$, then $B^-_{\delta_2} \subseteq B^-_{\delta_1}$. Indeed, if $x \in B^-_{\delta_2}$, then $x \in W^u_{\delta_2}(y)$ for some $y \in D^-_{\varepsilon,l}$. Since f is regular, D^-_ε has full measure, so $D^-_\varepsilon \cap W^u_{\delta_2}(y)$ has full conditional measure. So we can choose $y' \in W^u_{\delta_2}(y)$ with distance δ_1 from x along $W^u_{\delta_2}(y)$, giving us $x \in W^u_{\delta_1}(y')$, so $x \in B^-_{\delta_1}$.

In particular, if μ is an ergodic SRB measure that charges $B^-_{\delta_1}$, then since $B^-_{\delta_2} \subset B^-_{\delta_1}$, either $B^-_{\delta_2}$ is charged by μ, or has μ-measure 0. In the latter case, if there is an ergodic SRB measure μ_1 that charges $B^-_{\delta_2}$, then both μ_1 and μ are ergodic SRB measures charging $B^-_{\delta_1}$. To summarize, if $\delta_2 > \delta_1$, then $B^-_{\delta_1}$ is charged by at least as many, if not more, SRB measures as $B^-_{\delta_2}$. *A priori*, even though each B^-_δ only is charged by finitely many ergodic SRB measures by Lemma 3.2, the union as $\delta \to 0$ may be charged by infinitely many. The following lemma precludes this possibility.

Lemma 3.3. *Suppose the singular hyperbolic map $f : K \setminus N \to K$ satisfies (H1), (H3), and (H5)-(H9). There is a $\delta > 0$ such that for every ergodic SRB measure μ on Λ, $\mu(B^-_\delta) > 0$.*

Proof. Assumption (H5) states that N is composed of finitely many closed submanifolds with boundary. Call these submanifolds N_i. Observe that if U is a neighborhood of N, then $f(U)$ is a neighborhood of N^-. Because $f^j(N^-) \cap N = \emptyset$ for $1 \le j < k$ with $\lambda^k > 2$ by (H9), and because N^- and its images are closed (as is N), there is a radius $Q > 0$ so that the open neighborhoods $B_Q(N_i)$ of each N_i of radius Q are pairwise disjoint and whose first k images do not intersect N. Let $\delta < Q$.

Fix an $\varepsilon > 0$ and an ergodic SRB measure μ. Our regularity hypothesis (H3) implies $\mu(D^-_\varepsilon) > 0$. Therefore there is a generic point $x \in D^-_\varepsilon$, which implies there is an $r > 0$ and an $l \ge 1$ for which $\mu(D^-_{\varepsilon,l} \cap B_r(x)) > 0$. By virtue of the hyperbolic local product structure of D (see Proposition 2.7), there is an $\alpha > 0$ and a $\beta > 0$ for which $W^s_\beta(x) \subset B_r(x)$, and the local unstable leaves $W^u_\alpha(y) \subset B_r(x)$ are well-defined for every $y \in D^-_{\varepsilon,l} \cap W^s_\beta(x)$. Furthermore, on a subset $A \subset W^s_\beta(x)$ of full conditional measure, the leaves $W^u_\alpha(y)$ have positive conditional measure for every $y \in A$. Therefore, the set

$$R_1 = \bigcup_{y \in A} W^u_\alpha(y)$$

has positive μ-measure and is contained in $B_r(x)$.

Suppose $\alpha \ge \delta$. Let $y_0 \in A$. Then $W^u_\alpha(y_0)$ is the union of finitely many $W^u_\delta(y_i)$, with $y_i \in W^u_\alpha(y_0) \cap D^-_\varepsilon$. So by definition of B^-_δ, for $z \in W^u_\alpha(y_0)$, we may take $y = y_i$ in the definition of B^-_δ for one of the y_i's, so $z \in B^-_\delta$. So $W^u_\alpha(y_0) \subset B^-_\delta$ for every $y_0 \in A$. In particular, $R_1 \subset B^-_\delta$, so since R_1 has positive measure, $\mu(B^-_\delta) > 0$.

Now suppose $\alpha < \delta$, and let $x_1 = x$. By compactness of K, there is a time $j_1 \ge 1$ at which $f^{j_1}(W^u_\alpha(x_1))$ intersects N (and, by assumption, it does so transversally). For $1 \le j \le j_1$, the image $f^j(W^u_\alpha(x_1))$ is a local unstable leaf of size $\alpha_j \ge \lambda^j \alpha$. If $\lambda^j \alpha \ge \delta$ for some $j \le j_1$, then using the same arguments as in the previous paragraph, $f^j(W^u_\alpha(x_1)) \subset B^-_\delta$, for almost every $y_0 \in f^j(W^s_\beta(x_1))$, and more generally, $f^j(R) \subseteq B^-_\delta$, yielding the desired result.

On the other hand, if $\alpha_j < \delta$ for $1 \le j \le j_1$, then the leaf $f^{j_1}(W^u_\alpha(x_1))$ contains a ball in $W^u_{\mathrm{loc}}(f^{j_1}(x_1))$ of diameter $\alpha' \ge \frac{1}{2}\lambda^{j_1}\alpha$ that does not intersect N. Because μ is an SRB measure, the conditional measure on this ball is absolutely continuous with respect to the Riemannian measure—in particular, the set of generic points in this leaf has full μ-conditional measure. So choose a generic point x_2 in this ball. As with x_1, we can use the hyperbolic product structure induced by f to create a rectangle R_2 of positive μ-measure defined by

$$R_2 = \bigcup_{y\in A_2} W^u_{\alpha'}(y),$$

where $A_2 \subset W^s_{\beta_2}(x_2)$ is a set of full measure for some $\beta_2 > 0$. Relabel $\alpha_{j_1} = \alpha'$ to be the diameter of the local unstable leaf containing x_2 and not intersecting N. This leaf (and in fact all of R_2) lies inside $B_Q(N) = \bigcup_i B_Q(N_i)$, and by our construction of $B_Q(N)$, the first k images of this leaf under f will not intersect N. Therefore, either eventually one of the images of this leaf is of diameter $> \delta$, or this leaf intersects N. In the former case, as before, we have a rectangle of positive μ-measure lying inside of B^-_δ. In the latter case, the leaf is of diameter $\geq \lambda^k\alpha' \geq \frac{1}{2}\lambda^{k+j_1}\alpha$. Again, there is a ball in this leaf of diameter $\geq \frac{1}{2}\lambda^k\alpha' \geq \frac{1}{4}\lambda^{k+j_1}\alpha$ that does not intersect N. As before, we may find a generic point x_3 in this ball, and continue iterating this local leaf and resulting positive-measure rectangle.

Proceeding in this way, we construct a sequence of local unstable leaves $W^u_{\alpha_j}(x_m)$, $j = j_1+\cdots+j_{m-1}+l$, where each leaf image intersects N at time $j_1+\cdots+j_m$, and thus admits an open ball of sufficient size and not intersecting N. By construction, $j_{i+1} - j_i \geq k$ for each i, so each leaf has

$$\alpha_j = \alpha_{j_1+\ldots+j_{m-1}+l} \geq \frac{\lambda^j}{2^{m-1}}\alpha \geq \left(\frac{\lambda^k}{2}\right)^m \frac{\lambda^{j_1}}{2}\alpha.$$

As m increases, we eventually get $\left(\lambda^k/2\right)^m \lambda^{j_1}\alpha/2 \geq Q > \delta$ by (H9). Once this occurs, we have a rectangle of positive measure contained in B^-_δ. □

Proof of Theorem 3.1. Note $B^- := \bigcup_{\delta>0} B^-_\delta$ is invariant under f, and as we showed in Lemma 3.3, $\mu(B^-) > 0$ for every ergodic SRB measure μ. Therefore $\mu(B^-) = 1$ for every ergodic SRB measure μ, and since $\mu(D) = 1$ as well, every ergodic SRB measure gives full volume to $D \cap B^-$. If there are infinitely many ergodic SRB measures, then by Lemma 3.3, there is a $\delta > 0$ for which B^-_δ is charged by infinitely many SRB measures. But this contradicts Lemma 3.2. □

4. Examples

Maps described in 3.1 do exist, and as the following non-example will demonstrate, the hypotheses described in this result are necessary assumptions in general. Examples of singular hyperbolic attractors can be found in, for example, Lorenz-type attractors; but this class of singular hyperbolic maps includes cases where the singular set has countably many components, and admit countably many SRB measures.

4.1. **Lorenz-type attractors.** To begin, we describe the class of singular hyperbolic attractors generated by Lorenz-type maps of the unit square. The definition of these maps is as follows. Let $I = (-1,1)$, $K = I^2$, and $-1 = a_0 < a_1 < \cdots < a_m < a_{m+1} = 1$. Define the rectangles $P_i = I \times (a_i, a_{i+1})$ for $0 \leq i \leq m$, and $N = I \times \{a_0, \ldots, a_{m+1}\}$. Let $f : K \setminus N \to K$ be an injective map given by

$$f(x,y) = \big(\varphi(x,y),\, \psi(x,y)\big), \quad x, y \in I,$$

where the functions $\varphi, \psi : K \to \mathbb{R}$ satisfy the following conditions:

(L1) φ and ψ are continuous in $\overline{P}_i$ for each i, and:

$$\lim_{y\to a_i^+}\varphi(x,y)=\varphi_i^+,\quad \lim_{y\to a_i^+}\psi(x,y)=\psi_i^+,$$
$$\lim_{y\to a_i^-}\varphi(x,y)=\varphi_i^-,\quad \lim_{y\to a_i^-}\psi(x,y)=\psi_i^-,$$

where $\varphi_i^\pm$, $\psi_i^\pm$ do not depend on x for each i;

(L2) ψ and φ have two continuous derivatives in P_i. Furthermore, there are positive constants B_i^1, B_i^2, C_i^1, and C_i^2; constants $0\le\nu_i^1,\nu_i^2,\nu_i^3,\nu_i^4\le 1$; a sufficiently small constant $\gamma>0$; and continuous functions $A_i^1(x,y)$, $A_i^2(x,y)$, $D_i^1(x,y)$, and $D_i^2(x,y)$ that tend to zero uniformly over x as $y\to a_i$ or $y\to a_{i+1}$; so that for $(x,y)\in P_i$,

$$\left.\begin{array}{l} d\varphi(x,y)=B_i^1(y-a_i)^{-\nu_i^1}\left(1+A_i^1(x,y)\right)\\ d\psi(x,y)=C_i^1(y-a_i)^{-\nu_i^2}\left(1+D_i^1(x,y)\right)\end{array}\right\}\quad \text{if } y-a_i\le\gamma;$$

$$\left.\begin{array}{l} d\varphi(x,y)=B_i^2(a_{i+1}-y)^{-\nu_i^3}\left(1+A_i^2(x,y)\right)\\ d\psi(x,y)=C_i^2(a_{i+1}-y)^{-\nu_i^4}\left(1+D_i^2(x,y)\right)\end{array}\right\}\quad \text{if } a_{i+1}-y\le\gamma;$$

and additionally, $\|\varphi_{xx}\|,\|\psi_{xx}\|,\|\varphi_{xy}\|,\|\psi_{xy}\|\le$ const.;

(L3) the following inequalities hold:

$$\|f_x\|,\left\|g_y^{-1}\right\|<1;$$
$$1-\left\|g_y^{-1}\right\|\|f_x\|>2\sqrt{\left\|g_y^{-1}\right\|\|g_x\|\left\|g_y^{-1}f_y\right\|};$$
$$\left\|g_y^{-1}\right\|\|g_x\|<(1-\|f_x\|)\left(1-\left\|g_y^{-1}\right\|\right);$$

where $\|\cdot\|=\max_i\sup_{(x,y)\in P_i}|\cdot|$.

This class of maps includes the *geometric Lorenz attractor*, for which we have $m=1$, $a_1=0$, and

$$\varphi(x,y)=(-B|y|^{\nu_0}+Bx\operatorname{sgn}(y)|y|^{\nu}+1)\operatorname{sgn}(y),$$
$$\psi(x,y)=((1+A)|y|^{\nu_0}-A)\operatorname{sgn}(y),$$

where $0<A<1$, $0<B<\frac{1}{2}$, $1/(1+A)<\nu_0<1$, and $\nu>1$.

Theorem 4.1. *Let $f:I^2\setminus N\to I^2$ be a Lorenz-type map for which one of the following properties hold:*

(a) $\nu_i^j=0$, for $i=1,\dots,m$ and $j=1,2,3,4$;
(b) $\rho\left(f^n(\varphi_i^\pm,\psi_i^\pm),N\right)\ge C_ie^{-\gamma n}$ (where C_i are constants independent of n and $\gamma>0$ is sufficiently small).

Then f admits a singular hyperbolic attractor Λ, which is supported by at most finitely many ergodic SRB measures.

Remark 4.2. This result is also proven in [4], and is also a consequence of the arguments in both [2] and [14]. We present an additional proof of this result using Theorem 3.1 directly.

Proof. Properties (H1) and (H6)-(H9) are shown in [11], Theorem 17. The singular set N is the disjoint union of finitely many horizontal lines $I\times\{a_i\}$, $i=1,\dots,m$, so (H5) is satisfied. The statement now follows from Theorem 3.1. □

Remark 4.3. Condition (H6) is easy to verify for the geometric Lorenz attractor, as the map $\varphi : I^2 \setminus (I \times \{0\}) \to \mathbb{R}$ extends to ± 1 as $y \to 0$ from above or below. In particular, $N^- \cap K = \emptyset$, since the continuations of φ to N map N to the boundary of K, so (H6) is in fact trivial. More generally, this is true with any Lorenz-type attractor for which $\varphi_i^{\pm} = \pm 1$ or ∓ 1.

More generally, (H6) holds if (b) is satisfied in the statement of Theorem 4.1, provided $\gamma > 0$ is sufficiently small.

4.2. **The Belykh attractor.** We consider a map $f : K \setminus N \to K$, where $K = [-1, 1]^2$, and

$$N = \{(x, kx) \in K : -1 < x < 1\}$$

where $|k| < 1$. (More generally one can consider $N = \{(x, h(x)) : -1 < x < 1\}$ for a continuous function h.) We then choose constants $\lambda_1, \lambda_2, \mu_1, \mu_2$ so that

$$0 < \lambda_1, \mu_1 < \frac{1}{2} \quad \text{and} \quad 1 < \lambda_2, \mu_2 < \frac{2}{1 + |k|},$$

and define the map $T : K \setminus N \to \mathbb{R}^2$ by

$$T(x, y) = \begin{cases} (\lambda_1(x-1)+1,\ \lambda_2(y-1)+1) & \text{if } y > kx; \\ (\mu_1(x+1)-1,\ \mu_2(y+1)-1) & \text{if } y < kx. \end{cases}$$

This map was first introduced in [6] as a model of phase synchronization theory.

Theorem 4.4. *Define $T : K \setminus N \to \mathbb{R}^2$ as above.*

- *(a) The map T is a map from $K \setminus N$ into K, and satisfies conditions (H1) and (H5)-(H8).*
- *(b) For any choice of $\lambda_2 > 1$, and for all but countably many $\mu_2 > 1$ (the countably many exceptions depending on λ_2), there is a $\delta > 0$ so that T satisfies (H9) when $|k| < \delta$, and thus admits finitely many ergodic SRB measures for such k.*

Proof. The first of the above assertions is proven in [11]. To prove the second, first note that [11] shows that T satisfying (H5)-(H8) admits countably many SRB measures. In general, (H9) may fail; however, we will show that given $\lambda_2 > 0$, this can happen only for countably many choices of $\mu_2 > 0$. To see this, note that when $k = 0$, (H9) fails if the horizontal lines forming N^- lie inside $f^{-n}(N)$ for some $n > 0$, which only happens for countably many choices of μ_2. Given a pair λ_2 and μ_2 so that T satisfies (H9) with $k = 0$, the line segments forming N and $f^j(N^-)$ do not intersect for $0 \le j < k$, where $\max(\lambda_2, \mu_2)^k > 2$. Increasing $|k|$ will rotate these line segments; by continuity, if the change in $|k|$ is sufficiently small, these line segments will remain disjoint. So, for these choices of λ_2, μ_2, and k, T will admit finitely many SRB measures by Theorem 3.1. □

4.3. **Necessity of assumptions.** The singular set N may in principle have countably many components. If this is the case, then the attractor may admit infinitely many ergodic SRB measures, as the following example illustrates.

Let $P_k = (-1, 1) \times \left(2^{-k} - 1,\ 2^{-(k-1)} - 1\right)$ for $k \geq 0$. Then $K = I^2 = \overline{\bigcup_k P_k}$, and $N^1 := K \setminus \bigcup_k P_k$ is the countable union of line segments $(-1, 1) \times \{2^{-k} - 1\} =: N_k^1$.

Let $f : I^2 \setminus \left((-1, 1) \times \{0\}\right) \to I_2$ be the geometric Lorenz attractor, and let $f_k : P_k \setminus \left((-1, 1) \times \left\{\frac{2^{-k-1} + 2^{-k}}{2} - 1\right\}\right.$ be given by $f_k = h_k^{-1} \circ f \circ h_k$, where $h_k : P_k \to I^2$

is the conjugacy map given by

$$h_k(x,y) = \left(x, 2^{k+2}(y+1) - 3\right).$$

Now denote

$$\begin{aligned} N &= I^2 \setminus \left(P_k \setminus \left((-1,1) \times \left\{\frac{2^{-k-1}+2^{-k}}{2} - 1\right\}\right)\right) \\ &= (-1,1) \times \bigcup_{k\geq 0} \left(\left\{2^{-k} - 1, \frac{2^{-k}+2^{-k-1}}{2} - 1\right\}\right), \end{aligned}$$

and let $g : I^2 \setminus N \to I^2$ be given by

$$g(x,y) = f_k(x,y) \quad \text{for } (x,y) \in P_k \setminus \left((-1,1) \times \left\{\frac{2^{-k-1}+2^{-k}}{2} - 1\right\}\right).$$

Effectively, we have embedded the geometric Lorenz attractor into each disjoint rectangle P_k. The map g admits a singular hyperbolic attractor, and the singular set N is the disjoint union of countably many submanifolds. Since each orbit of g is entirely contained in one of the rectangles P_k, each P_k supports a distinct ergodic SRB measure. So the requirement that there are only finitely many components of the singular set N is a necessary assumption for our result to hold.

One may ask if a topologically transitive example of a singular hyperbolic attractor whose singular set has countably many components and with infinitely many ergodic SRB measures may be constructed. We are not aware of such an example. By Theorem 9 in [11], if the stable foliation of a singular hyperbolic map is locally continuous, then each component of topological transitivity is an ergodic component of an SRB measure. Therefore if the singular set N has infinitely many components, and (unlike the above Lorenz-type example) may be constructed so that the stable foliation is not locally continuous, it may admit infinitely many SRB measures and still be topologically transitive.

Problem 1. Does there exist a singular hyperbolic map $f : K \setminus N \to K$ violating condition (H8), whose restriction to its attractor Λ is transitive, and which admits infinitely many ergodic SRB measures?

Unlike the geometric Lorenz attractor, the stable leaves of the Belykh map may be arbitrarily small if $|k| > 0$. So if a similar construction to our "infinite Lorenz" example is applied to the Belykh attractor, the resulting stable foliation may fail to be locally continuous. It is shown in [11] that the Belykh attractor admits countably many components of topological transitivity; if the Belykh map can in fact be shown to be topologically transitive, such an infinite construction may solve the above problem.

Another approach to solving this problem would be to consider transitive singular hyperbolic attractors with multiple ergodic SRB measures. In view of [11], such a map would have to admit a stable foliation that fails to be locally continuous.

Problem 2. Does there exist a singular hyperbolic map $f : K \setminus N \to K$ whose restriction to its attractor Λ is transitive, and admits more than one ergodic SRB measure?

We do not know of such a map at this time. In this case, the Belykh attractor may not be a good candidate for consideration, as the stable foliation of the Belykh attractor is locally continuous.

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