Equilibrium States of Almost Anosov Diffeomorphisms

Penn State University Student Dynamics Seminar

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April 4 2019

Almost Anosov Diffeomorphisms

Definition

A C^4 diffeomorphism f on a Riemannian manifold M is *almost Anosov* if there exist two continuous families of cones $x \mapsto \mathcal{C}^u_x, \mathcal{C}^s_x \subset TM$ such that, except for a finite set S,

1. $Df_x\mathcal{C}^u_x \subseteq \mathcal{C}^u_{fx}$ and $Df_x\mathcal{C}^s_x \supseteq \mathcal{C}^s_{fx}$;
2. $\|Df_x v\| > \|v\| \ \forall v \in \mathcal{C}^u_x$ and $\|Df_x v\| < \|v\| \ \forall v \in \mathcal{C}^s_x$.

By continuity, it follows that for each $p \in S$,

- $Df_p\mathcal{C}^u_p \subseteq \mathcal{C}^u_p$ and $Df_p\mathcal{C}^s_p \supseteq \mathcal{C}^s_p$;
- $\|Df_p v\| \geq \|v\| \ \forall v \in \mathcal{C}^u_p$ and $\|Df_p v\| \leq \|v\| \ \forall v \in \mathcal{C}^s_p$.

Assume S is invariant, and in fact $fp = p$ for all $p \in S$ (by considering f^n).

Remark

The above definition will yield a regular Anosov diffeomorphism if we remove the clause "except for a finite set S".

Non-degeneracy

Denote $B_r(A) = \{x \in M : d(x, A) < r\}$, with $d(x, A)$ the Riemannian distance from x to the set $A \subset \mathbb{T}^2$.

Definition

An almost Anosov diffeomorphism is *non-degenerate* (up to third order) if there exist constants $r_0 > 0$ and κ^u, κ^s such that for all $x \in B_{r_0}(S)$,

$$\|Df_x v\| \geq \left(1 + \kappa^u d(x, S)^2\right) \|v\| \qquad \forall v \in \mathcal{C}_x^u,$$

$$\|Df_x v\| \leq \left(1 - \kappa^s d(x, S)^2\right) \|v\| \qquad \forall v \in \mathcal{C}_x^s.$$

If f is almost Anosov, then for any constant $r > 0$, there exist constants $0 < K^s < 1 < K^u$, depending on r, such that for all $x \notin B_r(S)$, and for all $v^u \in \mathcal{C}_x^u$ and $v^s \in \mathcal{C}_x^s$,

$$\|Df_x v\| \geq K^u \|v\| \qquad \text{and} \qquad \|Df_x v\| \leq K^s \|v\|$$

Stable and unstable submanifolds

Define the *local stable and unstable manifolds* at the point $x \in M$:

$$W^u_\varepsilon(x) = \left\{ y \in M : d\left(f^{-n}y, f^{-n}x\right) \leq \varepsilon \quad \forall n \geq 0 \right\},$$
$$W^s_\varepsilon(x) = \left\{ y \in M : d\left(f^{n}y, f^{n}x\right) \leq \varepsilon \quad \forall n \geq 0 \right\}.$$

Theorem (Hu 2000)

There exists an invariant decomposition of the tangent bundle into $TM = E^u \oplus E^s$ such that for every $x \in M$:

- $E^\eta_x \subseteq \mathcal{C}^\eta_x$ *for* $\eta = s, u$;
- $Df_x E^\eta_x = E^\eta(fx)$ *for* $\eta = s, u$;
- $W^\eta_\varepsilon(x)$ *is a* C^1 *curve, which is tangent to* $E^\eta(x)$ *for* $\eta = s, u$.

Furthermore, the decomposition $TM = E^u \oplus E^s$ is continuous everywhere except possibly on S.

Coordinates of Singularity

Proposition (Hu 2000)

If $f : M \to M$ is almost Anosov and $p \in S$, then $D^2 f_p = 0$, so there is a coordinate system around p for which f is expressible as

$$f(x,y) = \Big(\; x(1+\varphi(x,y)), \;\; y(1-\psi(x,y)) \; \Big), \qquad (1)$$

for $(x,y) \in \mathbb{R}^2$ and

$$\varphi(x,y) = a_0x^2 + a_1xy + a_2y^2 + O\left(|(x,y)|^3\right),$$
$$\psi(x,y) = b_0x^2 + b_1xy + b_2y^2 + O\left(|(x,y)|^3\right),$$

where $|(x,y)| := \sqrt{x^2+y^2}$ for $x,y \in \mathbb{R}$.

Assume $M = \mathbb{T}^2$, $f : \mathbb{T}^2 \to \mathbb{T}^2$ is almost Anosov with singular set $S = \{0\}$, and that $Df_0 = \mathrm{Id}$.

Almost Anosov Conjugacy

Assumption: There are constants r_0 and r_1, with $0 < r_0 < r_1$ for which the almost Anosov map $f : \mathbb{T}^2 \to \mathbb{T}^2$ is equal to a linear Anosov map $\widetilde{f} : \mathbb{T}^2 \to \mathbb{T}^2$ outside of $B_{r_1}(0)$, and within $B_{r_0}(0)$, f has the form (1).

Theorem (V.)

A nondegenerate almost Anosov diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ satisfying the above assumption is topologically conjugate to an Anosov diffeomorphism.

Corollary

Almost Anosov diffeomorphisms admit Markov partitions of arbitrarily small diameter.

Young diffeomorphisms: stable and unstable discs

An embedded C^1 disc $\gamma \subset M$ is an *unstable* (resp. *stable*) *disc* if for all $x, y \in \gamma$, we have $d\left(f^{-n}x, f^{-n}y\right) \to 0$ (resp. $d\left(f^n x, f^n y\right) \to 0$) as $n \to +\infty$.

Definition

A collection of embedded C^1 discs $\Gamma^u = \{\gamma^u\}$ is a *continuous family of unstable discs* if there is a homeomorphism $\Phi : K^s \times D^u \to \bigcup \gamma^u$, where $K^s \subseteq M$ is a Borel subset and $D^u \subset \mathbb{R}^d$ is the unit disc for some $d < \dim M$, satisfying:

- $\gamma^u = \Phi\left(\{x\} \times D^u\right)$ is an unstable disc;
- $x \mapsto \Phi|_{\{x\}\times D^u}$ is a continuous map from K^s to the space of C^1 embeddings of D^u into M that can be extended to a continuous map of $\overline{K^s}$.

Continuous families of stable discs are defined similarly.

Young diffeomorphisms: hyperbolic product structure

Definition

A set $\Lambda \subseteq M$ has *hyperbolic product structure* if there exists a continuous family $\Gamma^u = \{\gamma^u\}$ of unstable discs, and a continuous family of stable discs $\Gamma^s = \{\gamma^s\}$ such that

- $\dim \gamma^s + \dim \gamma^u = \dim M$;
- the γ^u discs intersect the γ^s discs at exactly one point transversally, with an angle uniformly bounded away from 0;
- $\Lambda = (\bigcup \gamma^u) \cap (\bigcup \gamma^s)$.

A subset $\Lambda_0 \subseteq \Lambda$ is an *s-subset* if it has hyperbolic product structure and is defined by the same family Γ^u of unstable discs as Λ, and a continuous subfamily of stable discs $\Gamma^s_0 \subseteq \Gamma^s$. A *u-subset* is defined similarly.

Young diffeomorphisms: definition

A map $f : M \to M$ is a *Young diffeomorphism* if:

1. There exists $\Lambda \subset M$ with hyperbolic product structure, a countable collection of continuous subfamilies $\Gamma_i^s \subset \Gamma^s$ of stable discs, and positive integers τ_i, $i \geq 0$, such that the s-subsets

$$\Lambda_i^s := \bigcup_{\gamma \in \Gamma_i^s} (\gamma \cap \Lambda) \subset \Lambda \tag{2}$$

are pairwise disjoint and satisfy

- *invariance*: for every $x \in \Lambda_i^s$,

$$f^{\tau_i}(\gamma^s(x)) \subset \gamma^s(f^{\tau_i}(x)), \quad f^{\tau_i}(\gamma^u(x)) \supset \gamma^u(f^{\tau_i}(x))$$

- *Markov property*: $\Lambda_i^u := f^{\tau_i}(\Lambda_i^s)$ is a u-subset of Λ such that

$$f^{-\tau_i}(\gamma^s(f^{\tau_i}(x)) \cap \Lambda_i^u) = \gamma^s(x) \cap \Lambda,$$
$$f^{\tau_i}(\gamma^u(x) \cap \Lambda_i^s) = \gamma^u(f^{\tau_i}(x)) \cap \Lambda$$

Young diffeomorphisms: definition

2. For every $\gamma^u \in \Gamma^u$, the leaf volume μ_{γ^u} on γ^u satisfies

$$\mu_{\gamma^u}\left(\gamma^u \cap \Lambda\right) > 0, \quad \mu_{\gamma^u}\left(\overline{\left(\Lambda \setminus \bigcup \Lambda_i^s\right) \cap \gamma^u}\right) = 0.$$

3. For $x \in \Lambda_i^s$, define $\tau(x) = \tau_i$ to be the inducing time, and the induced map $F : \bigcup_{i \in \mathbb{N}} \Lambda_i^s \to \Lambda$ by $F|_{\Lambda_i^s} = f^{\tau_i}|_{\Lambda_i^s}$. Then there is $0 < a < 1$ s.t. for any $i \in \mathbb{N}$, we have:
    - For $x \in \Lambda_i^s$, $y \in \gamma^s(x)$,

    $$d(F(x), F(y)) \leq a d(x, y);$$

    - For $x \in \Lambda_i^s$, $y \in \gamma^u(x) \cap \Lambda_i^s$,

    $$d(x, y) \leq a d(F(x), F(y)).$$

Young diffeomorphisms: definition

4. (*Bounded estimates of distortion*) Denote the subspace

$$E^u(f^k x) := T_{f^k x} f^k(\gamma^u(x)) = Df_x^k T_x \gamma^u(x),$$

and let $J^u F(x) = \det \left| DF|_{E_x^u} \right|$. There exists $c > 0$ and $\kappa \in (0,1)$ such that:

- For all $n \geq 0$, $x \in F^{-n}\left(\bigcup_{i \geq 1} \Lambda_i^s\right)$, and $y \in \gamma^s(x)$, we have
$$\left| \log \frac{J^u F(F^n(x))}{J^u F(F^n(y))} \right| \leq c\kappa^n.$$
- For any $i_0, \ldots, i_n \in \mathbb{N}$, $F^k(x), F^k(y) \in \Lambda_{i_k}^s$ for $0 \leq k \leq n$ and $y \in \gamma^u(x)$, we have
$$\left| \log \frac{J^u F(F^{n-k}(x))}{J^u F(F^{n-k}(y))} \right| \leq c\kappa^k.$$

5. There exists $\gamma^u \in \Gamma^u$ such that $\sum_{i=1}^{\infty} \tau_i \mu_{\gamma^u}(\Lambda_i^s \cap \gamma^u) = \int_{\gamma^u} \tau \, d\mu_{\gamma^u} < \infty$.

Equilibrium states and geometric potentials

Definition

Given a continuous potential function $\varphi : M \to \mathbb{R}$, a probability measure μ_φ on M is an *equilibrium measure* for φ if

$$P_f(\varphi) = h_{\mu_\varphi}(f) + \int_M \varphi \, d\mu_\varphi,$$

where $h_{\mu_\varphi}(f)$ is the metric entropy of (M, f) with respect to μ_φ, and $P_f(\varphi)$ is the topological pressure of φ; that is, $P_f(\varphi)$ is the supremum of $h_\mu(f) + \int_M \varphi \, d\mu$ over all f-invariant probability measures μ.

We consider equilibrium states of the *geometric t-potential*

$$\varphi_t(x) = -t \log \left| Df|_{E^u(x)} \right|.$$

We denote $\mu_t := \mu_{\varphi_t}$.

Decay of correlations and CLT

Definition

The map f has *exponential decay of correlations* with respect to a measure $\mu \in \mathcal{M}(f, M)$ and a class of functions $\mathcal{H}$ on M if there exists $\kappa \in (0,1)$ such that for any $h_1, h_2 \in \mathcal{H}$,

$$\left| \int (h_1 \circ f^n)\, h_2 \, d\mu - \int h_1 \, d\mu \int h_2 \, d\mu \right| \leq C\kappa^n$$

for some $C = C(h_1, h_2) > 0$. Furthermore, f satisfies the *Central Limit Theorem* (CLT) if for any $h \in \mathcal{H}$ that is not a coboundary (ie. $h \neq h' \circ f - h'$), there exists $\sigma > 0$ such that

$$\lim_{n\to\infty} \mu\left\{ \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left(h(f^i(x)) - \int h \, d\mu \right) < t \right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\tau^2/2\sigma^2} \, d\tau.$$

Thermodynamics of Young's diffeomorphisms

Recall that an *SRB measure* is a probability measure with positive Lyapunov exponents almost everywhere, and which has absolutely continuous conditional measures on unstable leaves.

Theorem (Pesin, Senti, Zhang 2016)

Let $f : M \to M$ be a $C^{1+\varepsilon}$ Young diffeomorphism of a compact Riemannian manifold M. Assume the inducing time τ is a first return time to Λ. Then the following hold:

1. *There is a unique equilibrium measure μ_1 for the potential φ_1, which is the unique SRB measure.*
2. *Suppose for some $C > 0$ and $h \in (0, h_{\mu_1}(f))$, we have $S_n \leq Ce^{hn}$, where S_n is the number of stable sets Λ_i^s with inducing time $\tau_i = n$. Then there is $t_0 < 0$ s.t. for every $t \in (t_0, 1)$, there is a measure $\mu_t \in \mathcal{M}(f, Y)$, where $Y := \left\{f^k(x) : x \in \bigcup \Lambda_i^s, 0 \leq k \leq \tau(x) - 1\right\}$, which is a unique equilibrium measure for φ_t.*

Thermodynamics of Young diffeomoprhisms

Theorem (continued)

3. *Suppose* $\gcd(\tau_i) = 1$, *and that there is* $K > 0$ *such that for every* $i \geq 0$, *every* $x, y \in \Lambda_i^s$, *and* $0 \leq j \leq \tau$,

$$d\left(f^j(x), f^j(y)\right) \leq K \max\left\{d(x,y), d(F(x), F(y))\right\}.$$

Then for every $t \in (t_0, 1)$, *the measure* μ_t *has exponential decay of correlations and satisfies the CLT with respect to a class of functions* $\mathcal{H}$ *which contains all Hölder continuous functions on* M.

Theorem (Shahidi, Zelerowicz 2018)

If the induced map is mixing, then the system is additionally Bernoulli.

Main result

Theorem (V.)

Given an almost Anosov map $f : \mathbb{T}^2 \to \mathbb{T}^2$ satisfying preceding assumption, the following statements hold:

1. *There is a $t_0 < 0$ so that for any $t \in (t_0, 1)$, there is a unique equilibrium measure μ_t associated to φ_t. This equilibrium measure has exponential decay of correlations and satisfies the central limit theorem with respect to a class of functions containing all Hölder continuous functions on $\mathbb{T}^2$. The map is mixing with respect to μ_t, and hence Bernoulli.*
2. *For $t = 1$, there are two equilibrium measures associated to φ_1: the Dirac measure δ_0 centered at the origin, and a unique invariant SRB measure μ. If f is Lebesgue-area preserving, this SRB measure coincides with Lebesgue measure.*
3. *For $t > 1$, δ_0 is the unique equilibrium measure associated to φ_t.*

Proof outline I

Step 1: Construct Young tower.

Let P be an element of the Markov partition for (M, f), and let $\tau(x)$ be the first return time of x to P. For $x \in P$, denote $\gamma^s(x)$ and $\gamma^u(x)$ respectively to be the connected component of the intersection of the stable and unstable leaves with P. For x with $\tau(x) < \infty$, define:

$$\Lambda^s(x) = \bigcup_{y \in U^u(x) \setminus A^u(x)} \gamma^s(y),$$

where $\widetilde{U}^u(x) \subseteq \widetilde{\gamma}^u(x)$ is an interval containing x, open in the induced topology of $\widetilde{\gamma}(x)$, and $\widetilde{A}^u(x) \subset \widetilde{U}^u(x)$ is the set of points that either lie on the boundary of the Markov partition, or never return to $\widetilde{P}$.

Proof outline II

Theorem (V.)

The collection of sets $\{\Lambda^s(x)\}$ *forms a countable collection* $\{\Lambda^s_i\}$ *of s-sets satisfying conditions (Y1) - (Y5), making* $f : M \to M$ *a Young's diffeomorphism with tower base*

$$\Lambda := \bigcup_{i=1}^{\infty} \overline{\Lambda^s_i}$$

- (Y1) follows from conjugacy to Anosov systems.
- (Y2) deals with measure-0 events and is easy to show.
- (Y3) follows because points on stable (resp. unstable) leaves do not expand (resp. contract) in the neighborhood of the singularity.
- (Y5) follows from Kac's theorem since τ is a first-return time.

Proof outline III

Condition (Y4) (bounded estimates of distortion) follows from the following result:

Theorem (Hu 2000)

There exists a constant $I > 0$ *and* $\theta \in (0,1)$ *such that if* $\gamma \subset f(B_{r_1}(0)) \setminus B_{r_1}(0)$ *is a* W^s*-segment (that is* γ *is a subset of a stable leaf, and is homeomorphic to an open interval in the induced topology), and if* $f^i(\gamma) \subset B_{r_1}(0)$ *for* $i = 1, \ldots, n-1$*, then for every* $x, y \in \gamma$*,*

$$\left| \log \frac{\left| Df^n|_{E^u(x)} \right|}{\left| Df^n|_{E^u(y)} \right|} \right| \leq I d^u(x,y)^\theta, \tag{3}$$

where $d^u(x,y)$ *is the induced Riemannian distance from* x *to* y *in the stable leaf* γ*.*

Now it's a straightforward calculation.

Proof outline IV

All that's left to show is

$$S_n := \#\{\Lambda_i^s : \tau_i = n\} \le Ce^{hn}$$

This follows from properties of the conjugate Anosov system $\widetilde{f} : \mathbb{T}^2 \to \mathbb{T}^2$ and the conjugacy, since $h_{\mathrm{top}}(\widetilde{f}) = h_m(\widetilde{f})$, where m is Lebesgue measure, and observation that

$$\left| \int \log |Df|_{E^u}| \, dm - \log \lambda \right| < \varepsilon$$

for r_1 sufficiently small.