Research Statement

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1 Introduction and Overview of Dynamical Systems

My general area of research is dynamical systems. Modern questions in dynamical systems began with the study of differential equations, investigating solutions to physical systems such as Newton's Three-Body Problem and models of atmospheric convection such as the Lorenz system. These physical problems have motivated a wider investigation into abstract discrete dynamical systems. A discrete dynamical system is a pair (X, f), in which X is a *phase space* (typically either a topological space, measure space, or smooth manifold), and $f : X \to X$ is a transformation law (typically a continuous, measurable, or smooth function as appropriate). Dynamical systems over subsets of smooth manifolds, treated as probability spaces, form the basis of *smooth ergodic theory*. One of the primary objects considered in smooth ergodic theory is the class of hyperbolic dynamical systems, such as Anosov diffeomorphisms, Smale horseshoes, and related stochastic systems (such as subshifts of finite type).

In my research, I have studied the statistical properties of two classes of nonuniformly hyperbolic dynamical systems:

(1) almost-Anosov diffeomorphisms of the torus [15], and

(2) smooth models of pseudo-Anosov homeomorphisms of surfaces of arbitrary genus [16].

I have also made progress on studying the ergodic properties of singular hyperbolic attractors in arbitrary dimensions. I plan to further investigate the statistical properties of these maps, such as decay of correlations. Additionally, I am interested in investigating the ergodic theory of other nonuniformly hyperbolic maps, such as flows admitting both hyperbolic and elliptic behavior at different points in the state space (see e.g. [4] and [13]).

2 Thermodynamics of Hyperbolic Systems

The primary tools used in my research in smooth ergodic theory come from thermodynamic formalism. Thermodynamic formalism uses the principles of statistical physics as a framework to study the ergodic properties of abstract dynamical systems and stochastic processes. In this section, I will give a brief overview of the questions investigated in smooth ergodic theory, and the ways in which thermodynamic formalism has been used to investigate the ergodic theory of hyperbolic systems.

One of the most well-known classes of hyperbolic dynamical systems is the class of Anosov diffeomorphisms. These are diffeomorphisms $f : M \to M$ of a Riemannian manifold M that are globally *uniformly hyperbolic*. In other words, they admit a continuous splitting of the tangent bundle into two distributions, $TM = E^u \oplus E^s$, for which we have $\lambda > 1$ such that:

$$|df_x^n v| \le \lambda^{-n} |v| \quad \text{for every } v \in E^s(x); \quad \text{and} \tag{1}$$

$$\left|df_x^{-n} v\right| \le \lambda^{-n} |v| \quad \text{for every } v \in E^u(x). \tag{2}$$

Given a continuous function $\varphi : M \to \mathbb{R}$ (called an *observable*) and a probability measure μ on M, the sequence of functions $(\varphi \circ f^n)_{n \geq 0}$ on M forms a stochastic process.

A natural question to ask is: given a smooth dynamical system $f : M \to M$, which probability measures μ on M are reasonable? To answer this question, we consider the points $x \in M$ for which the following limit exists:

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi\left(f^k(x)\right) =: \hat{\varphi}(x).$$

The function $\hat{\varphi}$ is the conditional expectation of φ with respect to the σ-algebra of f-invariant Borel subsets. If f is ergodic (that is, f-invariant Borel subsets all have either full or 0 measure), $\hat{\varphi}$ is constant and equal to the expectation $\int \varphi \, d\mu$. In this context, a probability measure μ on M may be considered "reasonable" if

$$\mu\left\{x \in M : \lim_{n \to +\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi\left(f^k(x)\right) = \int \varphi \, d\mu\right\} > 0. \tag{3}$$

Borel probability measures on M satisfying (3) are called *physical measures*. This comes from interpreting the sum as the average of a sequence of experimental observations of φ (the *time average*), and the integral as the actual expectation of φ (the *space average*).

In the 1970s, R. Bowen, D. Ruelle, and Y. Sinai described a large class of physical measures on uniformly hyperbolic dynamical systems. Their approach involved considering specific observables $\varphi \in C^0(M)$, and the *topological pressure* of the observable, defined by

$$P_f(\varphi) = \sup_{\mu \in \mathcal{M}_f} \left(h_\mu(f) + \int \varphi \, d\mu\right), \tag{4}$$

where $\mathcal{M}_f$ is the space of f-invariant Borel probability measures, and $h_\mu(f)$ is the metric entropy of f with respect to the measure μ. The sum $h_\mu(f) + \int \varphi \, d\mu$ is called *Helmholtz free energy* in statistical mechanics, and a probability distribution in a physical system that minimizes Helmholtz free energy is known as an *equilibrium state*[1]. This led to the first major class of problems in thermodynamic formalism:

Problem. Given a dynamical system $f : X \to X$ and an observable $\varphi : X \to \mathbb{R}$, does there exist an f-invariant probability measure μ on X that attains the supremum in (4)? And if such a measure exists, is it unique?

If μ is a Borel f-invariant probability measure that attains the supremum in (4), then μ is called an *equilibrium measure* or *equilibrium state*. One objective of thermodynamic formalism is to determine the existence and uniqueness of equilibrium states for a given dynamical system. This is an active area of research to this day for different systems, and a central problem in my own work.

In hyperbolic dynamics, at least two distinct equilibrium states for different observables are often discussed. For the observable $\varphi_0 \equiv 0$, an equilibrium state for φ_0 is a measure of maximal entropy for $f : X \to X$. On the other hand, the observable

[1] Helmholtz free energy is in fact a negative multiple of this sum. This is why in statistical physics an equilibrium state *minimizes* Helmholtz free energy, whereas in thermodynamic formalism, an equilibrium measure *maximizes* this sum.

$\varphi_1(x) = -\log\det\left|df_x|_{E^u(x)}\right|$ (known as the *geometric potential*) admits an equilibrium state known as a *Sinai-Ruelle-Bowen (SRB) measure* (see (2) for the definition of $E^u(x)$). SRB measures are physical measures that have absolutely continuous conditional measures on unstable submanifolds (which are the leaves of the foliation generated by the distribution E^u). For globally hyperbolic diffeomorphisms that preserve Riemannian volume, this volume is an SRB measure. On the other hand, for dissipative hyperbolic systems with attractors (such as the Smale-Williams solenoid), the Riemannian volume of the attractor is 0, so SRB measures are better suited to use as physical measures supported on the state space.

The above problem, therefore, is a generalization of the following two questions in smooth ergodic theory:

Problem. Suppose $f : X \to X$ is a hyperbolic dynamical system, X a compact subset of a Riemannian manifold M.

(a) Does there exist a unique measure of maximal entropy for f?

(b) Does there exist a unique SRB measure for f?

For uniformly hyperbolic diffemorphisms, such as Anosov and Axiom A diffeomorphisms, it is known that the answer to both of these questions is "yes" [2]. However, for several classes of dynamical systems exhibiting more general types of hyperbolic behavior (such as hyperbolic systems admitting singularities), the existence and uniqueness of SRB measures and measures of maximal entropy is an open problem.

Once a probability measure μ for the dynamical system $f : X \to X$ has been chosen, any class of functions $\mathcal{H}$ on X admits a collection of stochastic processes $(h \circ f^n)_{n\geq 0}$, $h \in \mathcal{H}$. A natural next step is to investigate the probability-theoretic properties of f. For example, the measurable system (X, f, μ) is said to satisfy the *Central Limit Theorem* with respect to $\mathcal{H}$ if for any $h \in \mathcal{H}$ that is not a coboundary (ie. $h \neq h' \circ f - h'$ for any $h' \in \mathcal{H}$), there exists $\sigma > 0$ such that

$$\lim_{n\to\infty} \mu\left\{\sqrt{n}\left(\frac{1}{n}\sum_{i=0}^{n-1} h(f^i(x) - \int h\, d\mu\right) < t\right\} = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{t} e^{-\tau^2/2\sigma^2}\, d\tau,$$

i.e. the differences between the Birkhoff averages and space average are asymptotically normally distributed with mean 0. Additionally, one may consider the *statistical correlations* between a random variable h_2 and a stochastic process $h_1 \circ f^n$ ($h_1, h_2 \in \mathcal{H}$).

3 Nonuniformly hyperbolic surface diffeomorphisms

The thermodynamics and ergodic theory of globally uniformly hyperbolic diffeomorphisms have been thoroughly investigated. In low dimensions, the statistical properties of many significant equilibrium measures (SRB measures and measures of maximal entropy in particular) are well established. However, the smooth ergodic theory of nonuniformly hyperbolic diffeomorphisms is still a rich area of active research.

In [3], M. Gerber and A. Katok provided an explicit procedure to construct nonuniformly hyperbolic diffeomorphisms on any surface. The construction is a perturbation of pseudo-Anosov homeomorphisms, which are locally uniformly hyperbolic surface homeomorphisms that are smooth except at finitely many singular fixed points. In the construction, the trajectories near these singularities are slowed down, so that the differentials become the identity on the tangent space. These pseudo-Anosov diffeomorphisms are C^∞ nonuniformly hyperbolic surface diffeomoprhisms. Unlike Anosov diffeomorphisms, which only exist on the torus in two dimensions, pseudo-Anosov diffeomorphisms may be constructed on any compact surface (as every compact surface admits pseudo-Anosov homeomorphisms).

In [16], we investigated the existence and uniqueness of equilibrium states for a family of observables for pseudo-Anosov diffeomorphisms. The observables I studied were the one-parameter family of *geometric t-potentials* defined by $\varphi_t(x) = -t \log\det \left| df_x|_{E^u(x)} \right|$. In particular, the equilibrium states for φ_0 and φ_1 include measures of maximal entropy and SRB measures, respectively. We considered the following questions:

Question 1. Given $t \in \mathbb{R}$, does the geometric t-potential $\varphi_t(x)$ for a pseudo-Anosov diffeomorphism $g : M \to M$ admit a unique equilibrium measure?

Question 2. If an equilibrium measure for $\varphi_t(x)$ exists, is there a class $\mathcal{H}$ of observables with respect to which (X, f, μ) satisfies the central limit theorem? Do correlations decay with respect to $\mathcal{H}$, and if so, how quickly?

Question 1 has a partially affirmative answer, and given a unique equilibrium state, the answer to Question 2 is yes. We were able to prove the following result:

Theorem 1 ([16]). *Let M be a compact Riemannian 2-manifold. Given any $t_0 < 0$, there is a pseudo-Anosov diffeomorphism $g : M \to M$ for which the following hold:*

1. *For any $t \in (t_0, 1)$, there is a unique equilibrium measure μ_t associated to φ_t. This equilibrium measure has exponential decay of correlations and satisfies the Central Limit Theorem with respect to a class of functions containing all Hölder continuous functions on M, and is Bernoulli. Additionally, the pressure function $t \mapsto P_g(\varphi_t)$ is real analytic in the open interval $(t_0, 1)$.*
2. *For $t = 1$, there are two classes of equilibrium measures associated to φ_1: convex combinations of Dirac measures concentrated at the singularities, and a unique invariant SRB measure μ.*
3. *For $t > 1$, the equilibrium measures associated to φ_t are precisely the convex combinations of Dirac measures concentrated at the singularities.*

Our proof relies on the technology of Young towers, which are a special case of Katukani-Rokhlin towers. The thermodynamics and ergodic properties of these dynamical towers are an active area of research [11, 14], and are central tools to my work in smooth ergodic theory generally.

Young towers were also used by Y. Pesin, S. Senti, and K. Zhang in [12] to prove a similar result about the Katok map of the torus. The Katok map is an early example of a nonuniformly hyperbolic surface diffeomorphism, originally described in [8]. Its construction is similar to the construction of the pseudo-Anosov diffeomorphisms described in [3], although the slow-down effect is applied to the fixed point of a hyperbolic toral automorphism at the origin of the torus. In [10], it was additionally shown that the Lebesgue measure μ_1 for the Katok map has polynomial decay of correlations. One question I am currently investigating is:

Question 3. Given a pseudo-Anosov diffeomorphism $g : M \to M$ of a compact Riemannian 2-manifold M, it is known that there is a unique SRB measure μ_1. What is the decay rate of the correlations for this SRB measure?

It is not known if the techniques in [10] can be applied directly to the pseudo-Anosov case; the slow-down procedure for the Katok map is generally speaking a more significant perturbation of a hyperbolic toral automorphism than the perturbation of a pseudo-Anosov homeomorphism to produce g. The technical arguments are therefore likely significantly different, and more investigation is required in order to study the statistical properties of this measure.

Other examples of nonuniformly hyperbolic surface diffeomorphisms include the class of *almost-Anosov diffeomorphisms*. These maps were first described in [6], where H. Hu and L-S. Young investigated the following general question:

Problem. Does a surface diffeomorphism exhibiting uniformly hyperbolic behavior "essentially everywhere" admit SRB measures?

The answer, as it turns out, is "not necessarily". In [6], H. Hu and L-S. Young constructed a diffeomorphism of the torus with uniform contraction along the stable leaves of the foliation, but with expansion that slowed down to 1 at a fixed point; in other words, the differential at the singular fixed point admitted eigenvalues of 1 and $\lambda < 1$. They showed that this "almost Anosov diffeomorphism" did not admit an SRB probability measure, but did admit an "infinite SRB measure", or an invariant measure with positive Lyapunov exponents almost everywhere, absolutely continuous conditional measures on the foliation, and giving finite mass to $M \setminus U$ for every neighborhood U of the singular fixed point.

Then in [5], H. Hu gave the following definition of an almost-Anosov map:

Definition 1. A map $f \in \text{Diff}^4(M)$ is *almost-Anosov* if there exist two families of cones $\mathcal{C}^u_x \subset T_xM$ and $\mathcal{C}^s_x \subset T_xM$ (with $x \mapsto \mathcal{C}^u_x$ and $x \mapsto \mathcal{C}^s_x$ continuous) such that, except at a finite set of fixed points,

- $df_x\mathcal{C}^u_x \subset \mathcal{C}^u_{f(x)}$ and $df_x\mathcal{C}^s_x \supset \mathcal{C}^s_{f(x)}$, and
- $|df_xv| > |v|$ for $v \in \mathcal{C}^u_x$, and $|df_xv| < |v|$ for $v \in \mathcal{C}^s_x$.

H. Hu considered almost-Anosov maps with a single "indifferent" fixed point, i.e. where the differential slowed down to the identity. His work showed that even when both expansion and contraction slow down to zero, there are examples of almost-Anosov maps admitting an SRB probability measure, and examples admitting an infinite SRB measure.

In [15], using thermodynamics of Young towers, we proved a similar result to Theorem 1, answering the following question:

Question 4. Does an almost-Anosov diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ of the torus admit a unique equilibrium measure for the geometric t-potentials $\varphi_t(x) = -t \log \det \left|df_x|_{E^u(x)}\right|$? If so, what are some of its statistical properties?

Specifically, we showed that in addition to the SRB measure, almost-Anosov diffeomorphisms admit unique measures of maximal entropy, which satisfy the Central Limit Theorem and have exponential decay of correlations. In fact, this is shown for a range of equilibrium measures associated to geometric t-potentials. This range does not include the SRB measure, however. In [7], H. Hu and X. Zhang used renewal theory to show that almost-Anosov diffeomorphisms have a polynomial upper and lower bound on the decay of correlations with respect to the SRB probability measure, giving further insight into the statistical properties of these maps.

I have also considered the thermodynamics of maps admitting both elliptic and hyperbolic behavior, such as the "elliptic island" systems described in [4] and [13].

Question 5. What are the statistical properties of maps admitting both elliptic and hyperbolic behavior?

By analyzing the slowed-down hyperbolic fixed point in these maps, one may be able to show that these maps admit Young towers. If so, this would be a step towards describing the statistical and ergodic properties of these maps with respect to different equilibrium measures, possibly including the symplectic measure described in [4].

4 Singular Hyperbolic Attractors

In addition to my work in thermodynamics of nonuniformly hyperbolic surface diffeomorphisms, I also study the ergodic properties of singular hyperbolic dynamical systems. These are dynamical systems whose underlying state space $K \subset M$ contains a set of singularities $N \subset K$ where the map $f : K \setminus N \to K$ fails to be continuous and/or fails to be differentiable. (Here, $K \subset M$ is a precompact open subset of a Riemannian manifold of arbitrary dimension.) A standard example of such a map is the geometric Lorenz attractor on the square $K = (-1, 1)^2$, for which the function is discontinuous on the line $y = 0$ (specifically the continuous extensions of this function to the rectangles $[-1, 1] \times [-1, 0]$ and $[-1, 1] \times [0, 1]$ each collapse this line to a single point). The Lozi attractor is another such example, in which the function f is continuous on the singular set, but not differentiable.

In [9] it is shown that in general, singular hyperbolic maps admit SRB measures, but these measuers may not be unique. The next natural question to ask is:

Question 6. How many distinct ergodic SRB measures does a singular hyperbolic attractor admit?

This is one of my current projects. Recently, we were able to show the following:

Theorem 2. *Let Λ be a singular hyperbolic attractor of a map $f : K \setminus N \to K$. Suppose the singular set N is a disjoint union of finitely many closed submanifolds with boundary. Then under certain regularity conditions, $f : \Lambda \to \Lambda$ admits finitely many ergodic components, and Λ admits finitely many distinct ergodic SRB measures.*

The assumption that N is a disjoint union of finitely many submanifolds is necessary. For example, one can consider the classic geometric Lorenz system $f : (-1, 1)^2 \setminus \big((-1, 1) \times \{0\}\big) \to (-1, 1)^2$, and embed countably many copies of this system into subsets of $(-1, 1)^2$ of the form $K_n = (-1, 1) \times (2^{-n} - 1, 2^{-(n-1)} - 1)$, $n \geq 0$. Each copy admits its own SRB measure. However, this example is not topologically transitive and not particularly informative.

Question 7. Does there exist a transitive singular hyperbolic attractor whose singular set has infinitely many connected components and admits infinitely many ergodic SRB measures?

Recently I have tried modifying the aforementioned example by partially extending the images of the Lorenz map copies into adjacent copies. Although such a map is topologically transitive, it is not known if it admits countably many ergodic SRB measures.

References

[1] L. Barreira and Y. Pesin. *Nonuniform Hyperbolicity: Dynamics of systems with nonzero Lyapunov exponents*, volume 115 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, 2007.

[2] R. Bowen. *Equilibrium states and ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics.* Springer, 1975.

[3] M. Gerber, A. Katok. Smooth models of pseudo-Anosov maps. *Ann. scient. Éc. Norm. Sup.* **4**(15):173-204, 1982.

[4] C. Liverani. Birth of an elliptic island in a chaotic sea. *Math Phys. Electron. J.* **10**, 2004.

[5] H. Hu. Conditions for the existence of SBR measures for "almost Anosov" diffeomorphisms. *Trans. Amer. Math. Soc.* **352**(5):2331-2367, 1999.

[6] H. Hu, L-S. Young. Nonexistence of SBR measures for some diffeomorphisms that are "almost Anosov". *Ergod. Th. Dyn. Sys* **15**:67-76, 1995.

[7] H. Hu, X. Zhang. Polynomial decay of correlations for almost Anosov diffeomorphisms. *Preprint.*

[8] A. Katok. Bernoulli diffeomorphisms on surfaces. *Ann. of Math. (2)*, **110**(3):529-547, 1979.

[9] Y. Pesin. Dynamical systems with generalized hyperbolic attractors: hyperbolic, ergodic and topological properties. *Ergod. Th. Dynam. Sys.* **12**:123-151, 1992.

[10] Y. Pesin, S. Senti, and F. Shahidi. Area preserving surface diffeomorphisms with polynomial decay rate are ubiquitous. *Preprint*, 2017.

[11] Y. Pesin, S. Senti, and K. Zhang. Thermodynamics of towers of hyperbolic type. *Trans. Amer. Math. Soc.* **368**(12):8519-8552. 2016.

[12] Y. Pesin, S. Senti, and K. Zhang. Thermodynamics of the Katok map (revised version). *Ergodic Theory Dynam. Systems*, **39**(3):764-794, 2019

[13] F. Przytycki. Examples of conservative diffeomorphisms of the two-dimensional torus with coexistence of elliptic and stochastic behavior. *Ergod. Th. Dyn. Sys.* **2**:439-463, 1982.

[14] F. Shahidi, A. Zelerowicz. Thermodynamics via inducing, *J. Stat. Phys*, **175**(2):351-383, 2019.

[15] D. Veconi. Equilibrium states of almost Anosov diffeomorphisms. *Disc. Cont. Dynam. Syst.* **40**(2):767-780, 2020.

[16] D. Veconi. Thermodynamics of smooth models of pseudo-Anosov homeomorphisms. To appear in *Erg. Th. Dyn. Sys.* 2021.