Equilibrium States of Almost Anosov Diffeomorphisms

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Almost Anosov Diffeomorphisms

Definition

A C^4 map f on a compact Riemannian surface M is an **almost Anosov diffeomorphism** (AAD) if there exist two continuous families of cones $x \mapsto \mathcal{C}^u_x, \mathcal{C}^s_x \subset TM$ such that, **except for a finite set** S,

1. $Df_x \mathcal{C}^u_x \subseteq \mathcal{C}^u_{fx}$ and $Df_x \mathcal{C}^s_x \supseteq \mathcal{C}^s_{fx}$;
2. $\|Df_x v\| > \|v\| \ \forall v \in \mathcal{C}^u_x$ and $\|Df_x v\| < \|v\| \ \forall v \in \mathcal{C}^s_x$.

By continuity, it follows that for each $p \in S$,

- $Df_p \mathcal{C}^u_p \subseteq \mathcal{C}^u_p$ and $Df_p \mathcal{C}^s_p \supseteq \mathcal{C}^s_p$;
- $\|Df_p v\| \geq \|v\| \ \forall v \in \mathcal{C}^u_p$ and $\|Df_p v\| \leq \|v\| \ \forall v \in \mathcal{C}^s_p$.

Assume $fp = p$ for all $p \in S$

Non-degeneracy

Let $B_r(A) = \{x \in M : d(x, A) < r\}$, $d(x, A)$ Riemannian distance from x to $A \subset M$.

Definition

An AAD is **non-degenerate** (up to third order) if there are $r_0 > 0$ and κ^u, κ^s s.t. for all $x \in B_{r_0}(S)$,

$$\|Df_x v\| \geq \left(1 + \kappa^u d(x, S)^2\right) \|v\| \qquad \forall v \in \mathcal{C}_x^u,$$

$$\|Df_x v\| \leq \left(1 - \kappa^s d(x, S)^2\right) \|v\| \qquad \forall v \in \mathcal{C}_x^s.$$

If f is AAD, then for any $r > 0$, there exist $0 < K^s < 1 < K^u$, depending on r, s.t. $\forall x \notin B_r(S)$, and $\forall v^u \in \mathcal{C}_x^u$ and $v^s \in \mathcal{C}_x^s$,

$$\|Df_x v\| \geq K^u \|v\| \qquad \text{and} \qquad \|Df_x v\| \leq K^s \|v\|$$

Stable and unstable submanifolds

Define the *local stable and unstable manifolds* at the point $x \in M$:

$$W^u_\varepsilon(x) = \left\{y \in M : d\left(f^{-n}y, f^{-n}x\right) \le \varepsilon \quad \forall n \ge 0\right\},$$
$$W^s_\varepsilon(x) = \left\{y \in M : d\left(f^{n}y, f^{n}x\right) \le \varepsilon \quad \forall n \ge 0\right\}.$$

Theorem (Hu 2000)

Let $f : M \to M$ be nondegenerate AAD. There exists an invariant decomposition of the tangent bundle $TM = E^u \oplus E^s$ s.t. $\forall x \in M$:

- $E^\eta_x \subseteq \mathcal{C}^\eta_x$ *for* $\eta = s, u$;
- $Df_x E^\eta_x = E^\eta_{fx}$ *for* $\eta = s, u$;
- $W^\eta_\varepsilon(x)$ *is a* C^1 *curve, which is tangent to* $E^\eta(x)$ *for* $\eta = s, u$.

Furthermore, the decomposition $TM = E^u \oplus E^s$ is continuous everywhere except possibly on S.

Example: Coordinates of Singularity

Proposition (Hu 2000)

If $f : M \to M$ is non-degenerate AAD and $p \in S$, then $D^2 f_p = 0$, so there is a coordinate system around p for which f is expressible as

$$f(x,y) = \Big(\; x(1+\varphi(x,y)), \;\; y(1-\psi(x,y)) \; \Big), \qquad (1)$$

for $(x,y) \in \mathbb{R}^2$ and

$$\varphi(x,y) = a_0 x^2 + a_1 xy + a_2 y^2 + O\left(|(x,y)|^3\right),$$
$$\psi(x,y) = b_0 x^2 + b_1 xy + b_2 y^2 + O\left(|(x,y)|^3\right),$$

where $|(x,y)| := \sqrt{x^2+y^2}$ for $x, y \in \mathbb{R}$.

Almost Anosov Conjugacy

Setting:

- $M = \mathbb{T}^2$, and $f : \mathbb{T}^2 \to \mathbb{T}^2$ is nondegenerate AAD with singular set $S = \{0\}$ and $Df_0 = \mathrm{Id}$.
- $\exists\, 0 < r_0 < r_1$ s.t. $f : \mathbb{T}^2 \to \mathbb{T}^2$ is a linear Anosov map $\widetilde{f} : \mathbb{T}^2 \to \mathbb{T}^2$ outside of $B_{r_1}(0)$, and within $B_{r_0}(0)$, f has the form (1).

Similar to Katok map, but must have $\|Df_x v\| < \|v\| \ \forall v \in \mathcal{C}^s_x$.

Theorem (V.)

A nondegenerate AAD $f : \mathbb{T}^2 \to \mathbb{T}^2$ satisfying the above assumptions is topologically conjugate to an Anosov diffeomorphism.

Corollary

Nondegenerate AADs satisfying the Assumption admit Markov partitions of arbitrarily small diameter.

Equilibrium states and geometric potentials

Definition

Let $\varphi : M \to \mathbb{R}$ be continuous. A probability measure μ_φ is an **equilibrium measure** for φ if

$$P_f(\varphi) = h_{\mu_\varphi}(f) + \int_M \varphi \, d\mu_\varphi,$$

where $h_{\mu_\varphi}(f)$ is the metric entropy of (M, f) and $P_f(\varphi)$ is the topological pressure of φ.

A probability measure μ is an **SRB measure** if it has positive Lyapunov exponents almost everywhere, and absolutely continuous conditional measures on unstable leaves.

We consider equilibrium states of the *geometric t-potential*

$$\varphi_t(x) = -t \log \left| Df|_{E^u(x)} \right|.$$

We denote $\mu_t := \mu_{\varphi_t}$.

Decay of correlations and CLT

Definition

- f has **exponential decay of correlations** with respect to a measure μ and a class of functions $\mathcal{H}$ on M if there exists $\kappa \in (0,1)$ s.t. for any $h_1, h_2 \in \mathcal{H}$,

$$\left| \int (h_1 \circ f^n)\, h_2 \, d\mu - \int h_1 \, d\mu \int h_2 \, d\mu \right| \leq C\kappa^n$$

  for some $C = C(h_1, h_2) > 0$.

- f satisfies the **Central Limit Theorem** (CLT) if for any $h \in \mathcal{H}$ s.t. $h \neq h' \circ f - h'$, $h' \in \mathcal{H}$, there is $\sigma > 0$ s.t.

$$\lim_{n\to\infty} \mu\left\{ \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \left(h(f^i(x)) - \int h\, d\mu \right) < t \right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\tau^2/2\sigma^2}\, d\tau.$$

Setting

- $M = \mathbb{T}^2$, and $f : \mathbb{T}^2 \to \mathbb{T}^2$ is nondegenerate AAD with singular set $S = \{0\}$ and $Df_0 = \mathrm{Id}$.
- $\exists\, 0 < r_0 < r_1$ s.t. $f : \mathbb{T}^2 \to \mathbb{T}^2$ is a linear Anosov map $\widetilde{f} : \mathbb{T}^2 \to \mathbb{T}^2$ outside of $B_{r_1}(0)$, and within $B_{r_0}(0)$, f has the form:

$$f(x, y) = \left(\; x(1 + \varphi(x, y)), \quad y(1 - \psi(x, y)) \; \right)$$

for $(x, y) \in \mathbb{R}^2$ and

$$\varphi(x, y) = a_0 x^2 + a_1 xy + a_2 y^2 + O\left(|(x, y)|^3\right),$$
$$\psi(x, y) = b_0 x^2 + b_1 xy + b_2 y^2 + O\left(|(x, y)|^3\right),$$

Main Result

Theorem (V.)

Given an almost Anosov map $f : \mathbb{T}^2 \to \mathbb{T}^2$ satisfying preceding assumption, the following statements hold:

1. *There is a $t_0 < 0$ so that for $t \in (t_0, 1)$, there is a unique equilibrium measure μ_t for φ_t. Further:*
    - *μ_t has exponential decay of correlations;*
    - *μ_t satisfies CLT with respect to a class of functions containing all Hölder functions;*
    - *the map is mixing with respect to μ_t.*
2. *For $t = 1$, there are two equilibrium measures associated to φ_1:*
    - *the Dirac measure δ_0 centered at the origin, and*
    - *a unique invariant SRB measure, which coincides w/ Lebesgue measure if f is area-preserving.*
3. *For $t > 1$, δ_0 is the unique equilibrium measure associated to φ_t.*

Thermodynamics of Young's diffeomorphisms

Theorem (Pesin, Senti, Zhang 2016; Shahidi, Zelerowicz 2018)

Let $f : M \to M$ be a $C^{1+\varepsilon}$ Young diffeomorphism of a compact Riemannian manifold M with base $\Lambda \subset M$, and $\tau = \tau(x) \in \mathbb{N}$ the first return time to Λ for $x \in \Lambda$.

1. *There is an equilibrium measure μ_1 for the potential φ_1, which is the unique SRB measure.*
2. *Let S_n be the number of sets $\Lambda_i^s \subset \Lambda$ with first return time $\tau_i = n$. Suppose for some $C > 0$ and $h \in (0, h_{\mu_1}(f))$, we have $S_n \leq Ce^{hn}$. Then there is $t_0 < 0$ s.t. for every $t \in (t_0, 1)$, there is a measure $\mu_t \in \mathcal{M}(f, Y)$, where $Y := \{f^k(x) : x \in \Lambda, 0 \leq k \leq \tau(x) - 1\}$, which is a unique equilibrium measure for φ_t.*

Thermodynamics of Young diffeomoprhisms

Theorem (continued)

3. *Suppose* $\gcd(\tau_i) = 1$*, and that there is* $K > 0$ *such that for every* $x, y \in \Lambda$ *w/ same first return time* τ*, and* $0 \le j \le \tau$*,*

$$d\left(f^j(x), f^j(y)\right) \le K \max\left\{d(x,y), d(F(x), F(y))\right\},$$

where $F(x) = f^{\tau(x)}(x)$*. Then for every* $t \in (t_0, 1)$*, the measure* μ_t *has exponential decay of correlations and satisfies the CLT with respect to a class of functions* $\mathcal{H}$ *which contains all Hölder continuous functions on* M*.*

4. *If the map* $f : M \to M$ *is mixing, then* (M, f, μ) *is Bernoulli.*

Young diffeomorphisms: definition

A map $f : M \to M$ is a *Young diffeomorphism* if:

1. There exists $\Lambda \subset M$ with hyperbolic product structure, a countable collection $\{\Gamma_i^s\}$ of continuous families of stable curves, and integers $\tau_i \geq 0$, $i \in \mathbb{N}$, s.t. the sets

$$\Lambda_i^s := \bigcup_{\gamma \in \Gamma_i^s} (\gamma \cap \Lambda) \subset \Lambda \tag{2}$$

are pairwise disjoint and satisfy

- *invariance*: for every $x \in \Lambda_i^s$,

$$f^{\tau_i}(\gamma^s(x)) \subset \gamma^s(f^{\tau_i}(x)), \quad f^{\tau_i}(\gamma^u(x)) \supset \gamma^u(f^{\tau_i}(x))$$

- *Markov property*: $\Lambda_i^u := f^{\tau_i}(\Lambda_i^s)$ satisfies

$$f^{-\tau_i}(\gamma^s(f^{\tau_i}(x)) \cap \Lambda_i^u) = \gamma^s(x) \cap \Lambda,$$
$$f^{\tau_i}(\gamma^u(x) \cap \Lambda_i^s) = \gamma^u(f^{\tau_i}(x)) \cap \Lambda$$

Young diffeomorphisms: definition

2. For every unstable leaf γ^u, the leaf volume μ_{γ^u} on γ^u satisfies

$$\mu_{\gamma^u}(\gamma^u \cap \Lambda) > 0, \quad \mu_{\gamma^u}\left(\overline{\left(\Lambda \setminus \bigcup \Lambda_i^s\right) \cap \gamma^u}\right) = 0.$$

3. There is $0 < \alpha < 1$ s.t. $\forall i \in \mathbb{N}$, we have:
   - For $x \in \Lambda_i^s$, $y \in \gamma^s(x)$,

     $$d(F(x), F(y)) \leq \alpha d(x, y);$$

   - For $x \in \Lambda_i^s$, $y \in \gamma^u(x) \cap \Lambda_i^s$,

     $$d(x, y) \leq \alpha d(F(x), F(y)).$$

Young diffeomorphisms: definition

4. (*Bounded estimates of distortion*) Let $\mathrm{Jac}^u(x) = \det \left| DF|_{E^u_x} \right|$. There exists $c > 0$ and $\kappa \in (0,1)$ such that:
   - For $n \geq 0$, $x \in \Lambda$, and $y \in \gamma^s(x)$,
   $$\left| \log \frac{\mathrm{Jac}^u(F^n(x))}{\mathrm{Jac}^u(F^n(y))} \right| \leq c\kappa^n.$$
   - For $i_0, \ldots, i_n \in \mathbb{N}$, and every $x, y \in \Lambda$ with $F^k(x),\ F^k(y) \in \Lambda^s_{i_k}$ for $0 \leq k \leq n$ and $y \in \gamma^u(x)$,
   $$\left| \log \frac{\mathrm{Jac}^u(F^{n-k}(x))}{\mathrm{Jac}^u(F^{n-k}(y))} \right| \leq c\kappa^k.$$
5. There exists an unstable leaf γ^u such that $\int_{\gamma^u} \tau \, d\mu_{\gamma^u} < \infty$.

Proof outline I

Construct Young tower:

- Let P be an element of the Markov partition for (M, f), and let $\tau(x)$ be the first return time of x to P.
- For $x \in P$, let $\gamma^s(x)$ and $\gamma^u(x)$ be the connected component of the intersection of the stable and unstable leaves with P.
- For x with $\tau(x) < \infty$, define:

$$\Lambda^s(x) = \bigcup_{y \in U^u(x) \setminus A^u(x)} \gamma^s(y),$$

where $U^u(x) \subseteq \gamma^u(x)$ is an interval containing x, open in the induced topology of $\gamma(x)$, and $A^u(x) \subset U^u(x)$ is the set of points that either lie on the boundary of the Markov partition, or never return to P.

Proof outline II

Theorem (V.)

The collection $\{\Lambda^s(x)\}$ *forms a countable collection* $\{\Lambda_i^s\}$ *of subsets of* P *satisfying conditions (Y1) - (Y5), making* $f : M \to M$ *a Young's diffeomorphism with tower base*

$$\Lambda := \bigcup_{i=1}^{\infty} \Lambda_i^s$$

- 1 (invariance/Markov property) follows from conjugacy to Anosov systems.
- 2 (positive unstable leaf volume) deals with measure-0 events and is easy to show.
- 3 (contraction/expansion) follows because points on stable (resp. unstable) leaves do not expand (resp. contract) in the neighborhood of the singularity.
- 5 (integrable return time) follows from Kac's theorem since τ is a first-return time.

Proof outline III

Condition 4 (bounded estimates of distortion) follows from the following result:

Theorem (Hu 2000)

Let f be a nondegenerate AAD. There exists a constant $I > 0$ and $\theta \in (0,1)$ such that if:

- $\gamma \subset f^{-1}(B_{r_1}(0)) \setminus B_{r_1}(0)$ *is a* W^s*-segment, and*
- $f^i(\gamma) \subset B_{r_1}(0)$ *for* $i = 1, \dots, n-1$,

then for every $x, y \in \gamma$,

$$\left| \log \frac{\left| Df^n|_{E^u(x)} \right|}{\left| Df^n|_{E^u(y)} \right|} \right| \leq I d^u(x,y)^\theta, \tag{3}$$

where $d^u(x,y)$ *is the induced Riemannian distance from* x *to* y *in the stable leaf* γ.

Proof outline IV

All that's left to show is:

- $\gcd(\tau_i) = 1$;
- there is $K > 0$ such that for every $x, y \in \Lambda$ with same first return time τ, and $0 \leq j \leq \tau$,

$$d\left(f^j(x), f^j(y)\right) \leq K \max\{d(x,y), d(F(x), F(y))\};$$

- $S_n := \#\{\Lambda_i^s : \tau_i = n\} \leq Ce^{hn}$.

This follows from properties of the conjugate Anosov system $\widetilde{f} : \mathbb{T}^2 \to \mathbb{T}^2$ and the conjugacy, since $h_{\mathrm{top}}(\widetilde{f}) = h_m(\widetilde{f})$, where m is Lebesgue measure, and observation that

$$\left| \int \log |Df|_{E^u}| \, dm - \log \lambda \right| < \varepsilon$$

for r_1 sufficiently small.