Boundary Value Problems for a Self-Adjoint Caputo Nabla Fractional Difference Equation

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Abstract

In this paper we develop the theory of initial and boundary value problems for a self-adjoint nabla fractional difference equation containing a Caputo fractional derivative that is given by

$$\nabla[p(t+1)\nabla^{\nu}_{a*}x(t+1)] + q(t)x(t) = h(t),$$

where $0 < \nu \le 1$. We begin by giving an introduction to the nabla fractional calculus and then look at a certain type of initial value problem containing the Caputo fractional derivative. We investigate properties of the specific self-adjoint nabla fractional difference equation given above, where we show existence and uniqueness for both initial and boundary value problems. We introduce the definition of a Cauchy function which allows us to solve initial value problems, as well as the definition of a Green's function that allows us to certain boundary value problems. Finally, we look at various inequalities regarding the Green's function for a particular self-adjoint boundary value problem where $p(t) = 1$, $q(t) = 0$, and $h(t) = 0$.

Background

Domain of $\mathbb{N}_a$

$$\mathbb{N}_a := \{a, a+1, a+2, \cdots\}.$$

Domain of $\mathbb{N}_a^b$

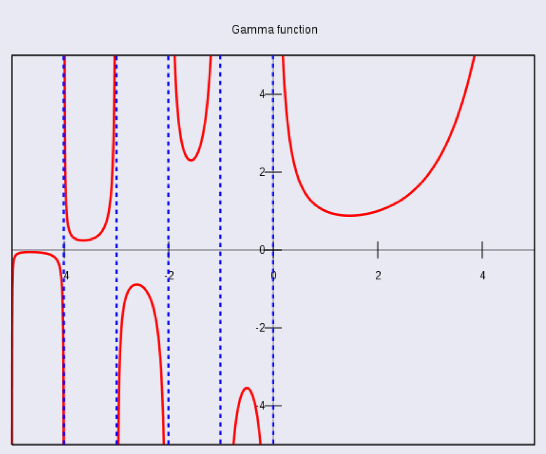
$$\mathbb{N}_a^b := \{a, a+1, a+2, \cdots, b-1, b\}, b-a \in \mathbb{N}_1.$$

Nabla Difference Operator

$$\nabla f(t) := f(t) - f(t-1), t \in \mathbb{N}_{a+1}.$$

Gamma Function

$$\Gamma(z) := \int_0^\infty e^{-t}t^{z-1}dt, z \in \mathbb{C}, \mathrm{Re}(z) > 0.$$



Rising Function

Let $f : \mathbb{N}_a \to \mathbb{R}$ and $a, b \in \mathbb{N}_a$ with $a < b$, the rising function is given by

$$\int_a^b f(t)\nabla t := \sum_{t=a+1}^{b} f(t).$$

Nabla Fractional Sum

Let $\nu > 0$ and $t \in \mathbb{N}_a$. Then:

$$\nabla_a^{-\nu}f(t) := \frac{1}{\Gamma(\nu)}\int_a^t (t-(s-1))^{\overline{\nu-1}}f(s).$$

Nabla Fractional Difference

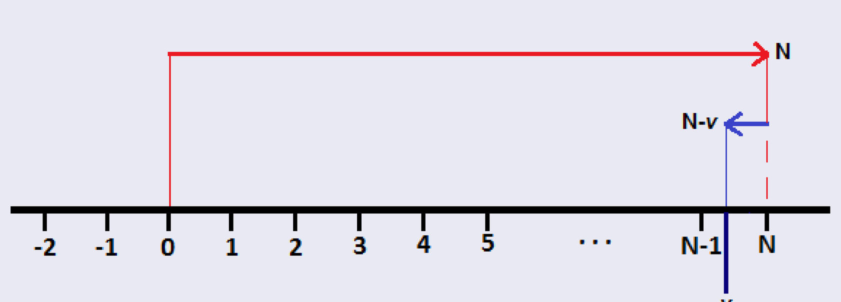
Let $\nu > 0$ and choose $N \in \mathbb{N}$ so that $N = \lceil \nu \rceil$. Then the ν-th order fractional difference of f is:

$$\nabla^\nu_a f(t) := \nabla^\nu_a f(t) := \nabla^N \nabla_a^{-(N-\nu)} f(t), t \in \mathbb{N}_{a+N}.$$

Caputo Fractional Difference

Let $\nu > 0$ and choose $N \in \mathbb{N}$ so that $N = \lceil \nu \rceil$. Then the ν-th order caputo fractional difference of f is

$$\nabla^\nu_{a*} f(t) := \nabla_a^{-(N-\nu)}\nabla^N f(t), t \in \mathbb{N}_{a-N+1}.$$



Continuity of the Nabla Fractional Difference

Suppose $f : \mathbb{N}_a \to \mathbb{R}$. Then the fractional difference $\nabla^\nu_a f$ is continuous with respect to $\nu \ge 0$.

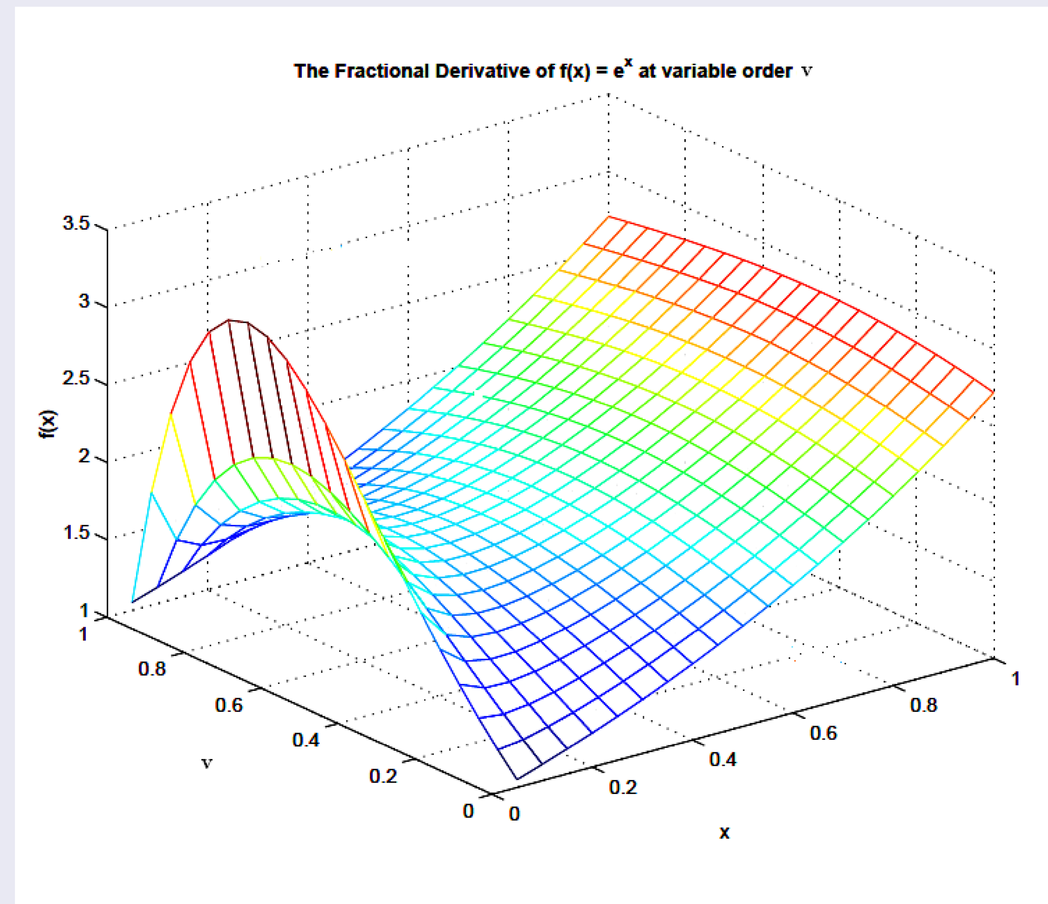


Figure : Continuous Analog of the Nabla Fractional Difference

Self-Adjoint Fractional Difference Equation and Operator

The self-adjoint Caputo fractional difference operator L_a is given by

$$L_a x := \nabla[p(t+1)\nabla^\nu_{a*}x(t+1)] + q(t)x(t), \quad t \in \mathbb{N}_{a+1},$$

for $x : \mathbb{N}_a \to \mathbb{R}$ and $p, q : \mathbb{N}_a \to \mathbb{R}$, and the self-adjoint Caputo fractional difference eqution is

$$L_a x = \nabla[p(t+1)\nabla^\nu_{a*}x(t+1)] + q(t)x(t) = h(t),$$

for $t \in \mathbb{N}_{a+1}$. Note L_a is a linear operator.

Cauchy Function

The Cauchy function $x : \mathbb{N}_a \times \mathbb{N}_a \to \mathbb{R}$ for $L_a x$ is the unique solution to the IVP

$$\begin{cases} L_s x(t,s) = \nabla[p(t+1)\nabla^\nu_{s*}x(t+1,s)] + q(t)x(t,s) = 0, \quad t \in \mathbb{N}_{a+1}, \\ x(s,s) = 0, \\ \nabla x(s+1,s) = \frac{1}{p(s+1)}, \end{cases}$$

for each fixed $s \in \mathbb{N}_a$.

IVP Variation of Constants

The unique solution to the self-adjoint IVP

$$\begin{cases} L_a y = h(t), \quad t \in \mathbb{N}_{a+1}, \\ y(a) = 0, \\ \nabla y(a+1) = 0, \end{cases}$$

is given by

$$y(t) = \int_a^t x(t,s)h(s)\nabla s,$$

where $x(t,s)$ is the Cauchy function of $L_a x$.

Boundary Value Problems

For $A, B \in \mathbb{R}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ satisfying $\alpha^2 + \beta^2 > 0$ and $\gamma^2 + \delta^2 > 0$, consider the boundary value problems (BVPs)

$$\begin{cases} L_a x = 0, \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ \alpha x(a) - \beta \nabla x(a+1) = 0, \\ \gamma x(b) + \delta \nabla x(b) = 0, \end{cases} \quad (H)$$

$$\begin{cases} L_a x = h(t), \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ \alpha x(a) - \beta \nabla x(a+1) = A, \\ \gamma x(b) + \delta \nabla x(b) = B. \end{cases} \quad (N)$$

Existence and Uniqueness: Suppose (H) has only the trivial solution. Then (N) has a unique solution.

Green's Function

The Green's function for *(H)*, $G : \mathbb{N}_a^b \times \mathbb{N}_a^b \to \mathbb{R}$ is given by

$$G(t,s) := \begin{cases} u(t,s), & \text{if} \quad a \le t \le s \le b, \\ v(t,s), & \text{if} \quad a \le s \le t \le b, \end{cases}$$

where $u(t,s)$ is the unique solution to the boundary value problem (BVP)

$$\begin{cases} L_a u = \nabla[p(t+1)\nabla^\nu_{a*}u(t+1,s)] + q(t)u(t,s) = 0, \quad t \in \mathbb{N}_{a+1}, \\ \alpha u(a,s) - \beta \nabla u(a+1,s) = 0, \\ \gamma u(b,s) + \delta \nabla u(b,s) = -\gamma x(b,s) - \delta \nabla x(b,s), \end{cases}$$

for each fixed $s \in \mathbb{N}_a$, where $x(t,s)$ is the Cauchy function for $L_a x$, and

$$v(t,s) := u(t,s) + x(t,s).$$

BVP Variation of Constants

If the BVP *(H)* has only the trivial solution, then for $A = B = 0$, the BVP *(N)* has the solution

$$y(t) = \int_a^b G(t,s)h(s)\nabla s,$$

where $G(t,s)$ is the Green's function.

Test for Uniqueness if $q(t) \equiv 0$

Define

$$\rho := \alpha\gamma\nabla_a^{-\nu}\frac{1}{p(b)} + \frac{\alpha\delta}{p(b)} + \frac{\beta\gamma}{p(a+1)}.$$

Then the BVP *(H)*, for $q(t) \equiv 0$, has only the trivial solution if and only if $\rho \ne 0$.

Inequalities of BVP Family Example

The Green's function for the boundary value problem

$$\begin{cases} \nabla[\nabla^\nu_{a*}x(t+1)] = 0, \quad t \in \mathbb{N}_{a+1}^{b-1}, \\ x(a) = 0, \\ x(b) = 0, \end{cases}$$

satisfies the inequalities

- $0 \ge G(t,s) \ge -\left(\frac{b-a}{4}\right)\left(\frac{\Gamma(b-a+1)}{\Gamma(\nu+1)\Gamma(b-a+\nu)}\right)$,
- $\int_a^b |G(t,s)|\,\nabla s \le \frac{(b-a)^2}{4\Gamma(\nu+2)}$,
- $\int_a^b |\nabla G(t,s)|\,\nabla s \le \frac{b-a}{\nu+1}$.

Remark on Symmetry

Unlike in the whole-order case, the Green's function of a fractional BVP is not always symmetric; in other words, $G(t,s) \ne G(s,t)$ for all $t, s \in \mathbb{N}_a^b$.

Example

Solve the BVP

$$\begin{cases} \nabla[\frac{1}{t+1}\nabla^\nu_{0*}y(t+1)] = 1, \quad t \in \mathbb{N}_1^6 \\ y(0) = 0, \quad y(7) = 0. \end{cases}$$

The corresponding homogeneous difference equation has only the trivial solution, so a unique solution exists. The Cauchy function for $\nabla[\frac{1}{t+1}\nabla^\nu_{0*}y(t+1)]$ is given by

$$x(t,s) = \frac{s(t-s)^{\overline{\nu}}}{\Gamma(\nu+1)} + \frac{(t-s)^{\overline{\nu+1}}}{\Gamma(\nu+2)}.$$

The Green's function is given by

$$G(t,s) := \begin{cases} u(t,s) = -\frac{t^{\overline{\nu+1}}}{7^{\overline{\nu+1}}}\left(\frac{s(7-s)^{\overline{\nu}}}{\Gamma(\nu+1)} + \frac{(7-s)^{\overline{\nu+1}}}{\Gamma(\nu+2)}\right), & \text{if } a \le t \le s \le b, \\ v(t,s) = u(t,s) + x(t,s), & \text{if } a \le s \le t \le b. \end{cases}$$

Therefore the solution is given by

$$\begin{aligned} y(t) &= \int_0^7 G(t,s)\nabla s = \int_0^t x(t,s)\nabla s + \int_0^7 u(t,s)\nabla s \\ &= \frac{2(t-1)^{\overline{\nu+2}}}{\Gamma(\nu+3)} + \frac{-12t^{\overline{\nu+1}}}{\Gamma(\nu+3)} \\ &= \frac{2(t-7)t^{\overline{\nu+1}}}{\Gamma(\nu+3)}. \end{aligned}$$