Equilibrium States of Almost Anosov Diffeomorphisms

2020 Vision for Dynamical Systems
In Memory of Anatole Katok
IMPAN – Będlewo,Poland

Dominic Veconi

Penn State University

August 15 2019

Almost Anosov Diffeomorphisms

A map f on a compact Riemannian surface M is an **almost Anosov diffeomorphism** (AAD) if there exist two continuous families of cones $x \mapsto \mathcal{C}^u_x, \mathcal{C}^s_x \subset TM$ such that, **except for a finite set S**,

1. $Df_x \mathcal{C}^u_x \subseteq \mathcal{C}^u_{fx}$ and $Df_x \mathcal{C}^s_x \supseteq \mathcal{C}^s_{fx}$;
2. $\|Df_x v\| > \|v\| \ \forall v \in \mathcal{C}^u_x$ and $\|Df_x v\| < \|v\| \ \forall v \in \mathcal{C}^s_x$.

By continuity, it follows that for each $p \in S$,

- $Df_p \mathcal{C}^u_p \subseteq \mathcal{C}^u_p$ and $Df_p \mathcal{C}^s_p \supseteq \mathcal{C}^s_p$;
- $\|Df_p v\| \geq \|v\| \ \forall v \in \mathcal{C}^u_p$ and $\|Df_p v\| \leq \|v\| \ \forall v \in \mathcal{C}^s_p$.

Assumptions:

- $f(p) = p$ and $Df_p = \mathrm{Id}$ for all $p \in S$;
- f is *nondegenerate* (quadratic bound on how quickly $Df_x \to \mathrm{Id}$ as $x \to S$).

Stable and unstable submanifolds

Define the *local stable and unstable manifolds* at the point $x \in M$:

$$W^u_\varepsilon(x) = \left\{y \in M : d\left(f^{-n}y, f^{-n}x\right) \leq \varepsilon \quad \forall n \geq 0\right\},$$
$$W^s_\varepsilon(x) = \left\{y \in M : d\left(f^{n}y, f^{n}x\right) \leq \varepsilon \quad \forall n \geq 0\right\}.$$

Theorem (Hu 2000)

Let $f : M \to M$ be nondegenerate AAD. There exists an invariant decomposition of the tangent bundle $TM = E^u \oplus E^s$ s.t. $\forall x \in M$:

- $E^\eta_x \subseteq \mathcal{C}^\eta_x$ *for* $\eta = s, u$;
- $Df_x E^\eta_x = E^\eta_{fx}$ *for* $\eta = s, u$;
- $W^\eta_\varepsilon(x)$ *is a* C^1 *curve, which is tangent to* $E^\eta(x)$ *for* $\eta = s, u$.

Furthermore, the decomposition $TM = E^u \oplus E^s$ is continuous everywhere except possibly on S.

SRB measures of AADs

- An SRB measure is a probability measure μ on a manifold M with positive Lyapunov exponents almost everywhere and absolutely continuous conditional measures on unstable leaves (w.r.t. Lebesgue).
- An *infinite* SRB measure μ is similar, but $\mu(M) = \infty$ and for any open $U \supset S$, $\mu(M \setminus U) < \infty$.

Theorem (Hu, Young 1995)

If $f : M \to M$ is a transitive AAD with singular fixed point $p \in M$ and $Df_p = \mathrm{diag}(\lambda, 1)$, $0 < \lambda < 1$, then f admits an infinite SRB measure.

Theorem (Hu 2000)

If a transitive AAD $f : M \to M$ has singular fixed point $p \in M$ with $Df_p = \mathrm{Id}$, then f will admit either an SRB probability measures or infinite SRB measure.

Example: Coordinates of Singularity

Theorem (Hu 2000)

If $f : M \to M$ is non-degenerate AAD and $p \in S$ has $Df_p = \mathrm{Id}$, then $D^2 f_p = 0$, so there is a coordinate system around p for which f is expressible as

$$f(x,y) = \Big(\; x\big(1+\varphi(x,y)\big), \;\; y\big(1-\psi(x,y)\big) \; \Big), \tag{1}$$

for $(x,y) \in \mathbb{R}^2$ and

$$\varphi(x,y) = a_0 x^2 + a_1 xy + a_2 y^2 + O\left(|(x,y)|^3\right),$$
$$\psi(x,y) = b_0 x^2 + b_1 xy + b_2 y^2 + O\left(|(x,y)|^3\right),$$

where $|(x,y)| := \sqrt{x^2+y^2}$ for $x, y \in \mathbb{R}$.

Almost Anosov Conjugacy

Setting:

- $M = \mathbb{T}^2$, and $f : \mathbb{T}^2 \to \mathbb{T}^2$ is transitive nondegenerate AAD with singular set $S = \{0\}$ and $Df_0 = \mathrm{Id}$.
- $\exists\, 0 < r_0 < r_1$ s.t. $f : \mathbb{T}^2 \to \mathbb{T}^2$ is a linear Anosov map $\widetilde{f} : \mathbb{T}^2 \to \mathbb{T}^2$ outside of $B_{r_1}(0)$, and within $B_{r_0}(0)$, f has the form (1).

Theorem (V. 2019)

A nondegenerate AAD $f : \mathbb{T}^2 \to \mathbb{T}^2$ satisfying the above assumptions is topologically conjugate to an Anosov diffeomorphism.

Corollary

Nondegenerate AADs satisfying the Assumption admit Markov partitions of arbitrarily small diameter.

Equilibrium states and geometric potentials

Let $\varphi : M \to \mathbb{R}$ be continuous. A probability measure μ_φ is an **equilibrium measure** for φ if

$$P_f(\varphi) = h_{\mu_\varphi}(f) + \int_M \varphi \, d\mu_\varphi,$$

where $h_{\mu_\varphi}(f)$ is the metric entropy of (M, f) and $P_f(\varphi)$ is the topological pressure of φ.
We consider equilibrium states of the *geometric t-potential*

$$\varphi_t(x) = -t \log \left| Df|_{E^u(x)} \right|.$$

We denote $\mu_t := \mu_{\varphi_t}$.

Decay of correlations and CLT

- f has **exponential decay of correlations** with respect to a measure μ and a class of functions $\mathcal{H}$ on M if there exists $\kappa \in (0,1)$ s.t. for any $h_1, h_2 \in \mathcal{H}$,

$$\left| \int (h_1 \circ f^n)\, h_2 \, d\mu - \int h_1 \, d\mu \int h_2 \, d\mu \right| \leq C\kappa^n$$

  for some $C = C(h_1, h_2) > 0$.

- f satisfies the **Central Limit Theorem** (CLT) if for any $h \in \mathcal{H}$ s.t. $h \neq h' \circ f - h'$, $h' \in \mathcal{H}$, there is $\sigma > 0$ s.t.

$$\lim_{n\to\infty} \mu\left\{ \sqrt{n}\left(\frac{1}{n} S_n(h) - E(h) \right) < t \right\} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\tau^2/2\sigma^2}\, d\tau$$

  where $S_n(h) = \sum_{i=0}^{n-1} h(f^i(x))$ and $E(h) = \int_M h \, d\mu$.

Setting

- $M = \mathbb{T}^2$, and $f : \mathbb{T}^2 \to \mathbb{T}^2$ is transitive nondegenerate AAD with singular set $S = \{0\}$ and $Df_0 = \mathrm{Id}$.
- $\exists\, 0 < r_0 < r_1$ s.t. $f : \mathbb{T}^2 \to \mathbb{T}^2$ is a linear Anosov map $\widetilde{f} : \mathbb{T}^2 \to \mathbb{T}^2$ outside of $B_{r_1}(0)$, and within $B_{r_0}(0)$, f has the form:

$$f(x,y) = \Big(\; x(1 + \varphi(x,y)), \quad y(1 - \psi(x,y)) \; \Big)$$

for $(x,y) \in \mathbb{R}^2$ and

$$\varphi(x,y) = a_0 x^2 + a_1 xy + a_2 y^2 + O\left(|(x,y)|^3\right),$$
$$\psi(x,y) = b_0 x^2 + b_1 xy + b_2 y^2 + O\left(|(x,y)|^3\right),$$

Main Result

Theorem (V. 2019)

Given an almost Anosov map $f : \mathbb{T}^2 \to \mathbb{T}^2$ satisfying preceding assumption, the following statements hold:

1. *There is a $t_0 < 0$ s.t. for $t \in (t_0, 1)$, there is a unique equilibrium measure μ_t for φ_t. Further:*
   - *μ_t satisfies CLT with respect to a class of functions containing all Hölder functions;*
   - *μ_t has exponential decay of correlations with respect to this class of functions, and is hence mixing;*
   - *the map is Bernoulli with respect to μ_t.*
2. *For $t = 1$, there are two equilibrium measures associated to φ_1:*
   - *the Dirac measure δ_0 centered at the origin, and*
   - *a unique invariant SRB measure, which coincides w/ Lebesgue measure if f is area-preserving.*
3. *For $t > 1$, δ_0 is the unique equilibrium measure associated to φ_t.*

Young diffeomorphisms (general idea)

The proof of the main result relies on the technology of *Young towers*.

Given $f : M \to M$ and $\Lambda \subset M$, let $\tau : \Lambda \to \mathbb{N}$ be an *inducing time* (often first-return time) and let $F = f^\tau : \Lambda \to \Lambda$ be the *induced map*, defined by $F(x) = f^{\tau(x)}(x)$.

The map $f : M \to M$ is a *Young diffeomorphism* with base $\Lambda \subset M$ if Λ has hyperbolic product structure, and F satisfies certain "nice" properties, including:

- Stable (resp. unstable) leaves are invariant under F (resp. F^{-1});
- F (resp. F^{-1}) contracts points in the same stable (resp. unstable) leaf as $n \to \infty$ (resp. $n \to -\infty$)
- τ is integrable on some unstable leaf;
- Distortion estimates are bounded (more on this later).

Thermodynamics of Young's diffeomorphisms

Let $f : M \to M$ be a $C^{1+\varepsilon}$ Young diffeomorphism of a compact Riemannian manifold M with base $\Lambda \subset M$ and first return time $\tau : \Lambda \to \mathbb{N}$. Under certain arithmetic and combinatorial conditions:

Theorem (Pesin, Senti, Zhang 2016)

- *$\exists$ an equilibrium measure μ_1 for the potential φ_1, which is the unique SRB measure;*
- *$\exists$ $t_0 < 0$ s.t. for $t \in (t_0, 1)$, there is a unique equilibrium measure μ_t for φ_t on $Y := \left\{ f^k(x) : x \in \Lambda,\, 0 \le k \le \tau(x) - 1 \right\}$;*
- *The measure μ_t has exponential decay of correlations and satisfies the CLT with respect to a class of functions $\mathcal{H}$ which contains all Hölder functions on M.*

Theorem (Shahidi, Zelerowicz 2018)

If $f : M \to M$ is mixing, then (M, f, μ_t) is Bernoulli.

Constructing Tower

Since AAD (M, f) is topologically conjugate to Anosov system, let $\mathcal{P}$ be a Markov partition for f.

- Let $P \in \mathcal{P}$, and let $\tau(x)$ be first return time of x to P.
- For $x \in P$, let $\gamma^s(x)$ and $\gamma^u(x)$ be the connected component of the intersection of the stable and unstable leaves with P.
- For x with $\tau(x) < \infty$, define the "stable strips":

$$\Lambda^s(x) = \bigcup_{y \in U^u(x) \setminus A^u(x)} \gamma^s(y),$$

  where $U^u(x) \subseteq \gamma^u(x)$ is an open interval containing x, and

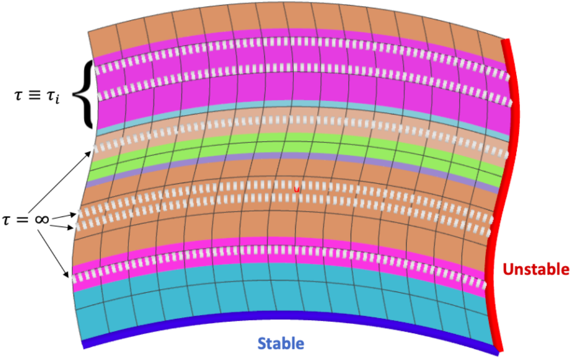
$$A^u(x) = \{y \in U^u(y) : y \in \partial P \text{ or } \tau(y) = \infty\}$$

Constructing Tower

Assume $U^u(x)$ is small enough s.t. $\tau|_{\Lambda^s(x)} \equiv \mathrm{const}\ \forall x \in P$ w/ $\tau(x) < \infty$. We get countable collection $\{\Lambda_i^s\}_{i\geq 1}$ w/ $\tau|_{\Lambda_i^s} \equiv \tau_i \in \mathbb{N}$. Define $\Lambda = \bigcup_{i\geq 1} \Lambda_i^s$.

Theorem (V. 2019)

The AAD $f : M \to M$ is a Young's diffeomorphism with tower base Λ.



Bounded Distortion

Most properties of Young diffeomorphism are easy to check, and follow from corresponding properties of Anosov diffeomorphisms. The one tricky property is *bounded distortion:*

There exist $c > 0$ and $\kappa \in (0,1)$ such that:

1. For all $n \geq 0$, $x \in \Lambda$ and $y \in \gamma^s(x)$, we have
$$\left| \log \frac{\left| DF|_{E^u(F^n(x))} \right|}{\left| DF|_{E^u(F^n(y))} \right|} \right| \leq c\kappa^n;$$
2. For all $n \geq 0$, and $F^k(x), F^k(y) \in \Lambda$ for $0 \leq k \leq n$ and $y \in \gamma^u(x)$, we have
$$\left| \log \frac{\left| DF|_{E^u(F^{n-k}(x))} \right|}{\left| DF|_{E^u(F^{n-k}(y))} \right|} \right| \leq c\kappa^k.$$

Bounded Distortion

This property follows from the following result:

Theorem (Hu 2000)

Let f be a nondegenerate AAD. There exists a constant $I > 0$ and $\theta \in (0,1)$ such that if:

- *x, y lie in the same stable leaf outside slowdown neighborhood $B_{r_1}(0)$ of singularity, and*
- *$f^i(x), f^i(y) \in B_{r_1}(0)$ for $i = 1, \dots, n-1$,*

then:

$$\left| \log \frac{\left| Df^n|_{E^u(x)} \right|}{\left| Df^n|_{E^u(y)} \right|} \right| \leq I d^u(x,y)^\theta, \tag{2}$$

where $d^u(x,y)$ is the induced Riemannian distance from x to y in the stable leaf γ.

Equilibrium state existence

- Using previous results, this gives us a unique equilibrium measure μ_t for $t < 1$ on the set

$$Y := \left\{ f^k(x) : x \in \Lambda, 0 \le k \le \tau(x) - 1 \right\}$$

- If $\hat{P}$ is another element of the Markov partition for (M, f), same argument gives us unique equilibrium measure $\hat{\mu}_t$ for $t < 1$ and corresponding set $\hat{Y}$.
- Assuming (M, f) is topologically transitive, since $\mu_t(U) > 0$ and $\hat{\mu}_t(\hat{U}) > 0$ for every open $U \supset P$, $\hat{U} \supset \hat{P}$, and $f^k(U) \cap \hat{U} \neq \varnothing$ for some $k \geq 1$, it follows from uniqueness that $\mu_t = \hat{\mu}_t$.

Thank you!