SRB Measures of Almost Anosov Diffeomorphisms
Dynamics Working Seminar

1 Definitions and main results

M is a C^∞ 2-dim. compact Riemannian manifold ($\partial M = \varnothing$), $m = dV_g$.

Definition 1. Let f be a C^r diffeomorphism, $r > 1$. A compact invariant set $\Lambda \subseteq M$ is *almost hyperbolic* if $\exists$ two continuous invariant families of cones $x \mapsto \mathcal{C}^u_x, \mathcal{C}^s_x$ of the tangent bundle $T_\Lambda M$ such that, except for a finite set $S \subseteq \Lambda$,

- $Df_x \mathcal{C}^u_x \subseteq \mathcal{C}^u_{fx}$ and $Df_x \mathcal{C}^s_x \supseteq \mathcal{C}^u_{fx}$;
- $|Df_x(v)| \geq |v|\ \forall\ v \in \mathcal{C}^u_x$ and $|Df_x(v)| \leq |v|\ \forall\ v \in \mathcal{C}^s_x$.

If $\Lambda = M$, then f is *almost Anosov*.

We assume that S is invariant, and consists of fixed points. In most examples of almost Anosov systems, we further assume $S = \{p\}$.

Example 1. The Katok map is an almost Anosov diffeomorphism.

It follows from continuity of the cone decomposition that f is uniformly hyperbolic away from p, ie. for any $r > 0$, there are constants $K^s \in (0, 1)$ and $K^u \geq 1$ so that, for $x \notin B(S, r)$,

$$|Df_x v| \geq K^u |v| \quad \forall v \in \mathcal{C}^u_x,$$
$$|Df_x v| \leq K^s |v| \quad \forall v \in \mathcal{C}^s_x$$

However, it is possible that $|Df_x v|/|v| \to 1$ as $x \to p$. We want to control the speed at which this happens.

Definition 2. An almost Anosov diffeomorphism f is *nondegenerate* if there are constants $r_0 > 0$ and $\kappa^u, \kappa^s > 0$ so that for $x \in B(S, r_0)$,

$$|Df_x v| \geq \left(1 + \kappa^u d(x, S)^2\right) |v| \quad \forall v \in \mathcal{C}^u_x,$$
$$|Df_x v| \leq \left(1 - \kappa^s d(x, S)^2\right) |v| \quad \forall v \in \mathcal{C}^s_x$$

Broad question in nonuniform hyperbolicity and thermodynamics thereof: whether certain maps (or classes of maps) admit SRB measures.

In a way, almost Anosov systems are the "most mildly nonuniform" maps, so they're a good class of maps to construct examples and identify specific cases for. That said, the thermodynamics can be quite complicated (as we know from studying the Katok map).

Example 2. Suppose we have a topologically transitive diffeomorphism f on a Riemannian 2-manifold M with a fixed point p so that:

- there is a constant $K^s < 1$ and a continuous function K^u so that $K^u(p) = 1$ and $K^u(x) > 1$ for $x \neq 0$, and
- a decomposition $T_x M = E^u_x \oplus E^s_x$ at every $x \in M$ so that

  $$|Df_x(v)| \leq K^s |v| \quad \forall v \in E^s_x,$$
  $$|Df_x(v)| \geq K^u(x) |v| \quad \forall v \in E^u_x,$$

  and $|Df_p(v)| = |v|$ for $v \in E^u_p$.

(e.g. you could take a hyperbolic toral automorphism and deform it near the origin.) In this case, f does not admit an SRB measure. However, it does admit an *infinite* measure μ with positive Lyapunov exponents μ-a.e., absolutely continuous conditional measures on (weak) unstable manifolds, and $\mu(M \setminus U) < \infty$ for every open neighborhood U of the origin.

We call such a measure an *infinite SRB measure.* (Without qualification, "SRB measure" means a probability measure.) Allowing this extension, we are able to make some broad statements about thermodynamics of almost Anosov diffeomorphisms with *indifferent* fixed points, that is, fixed points with $Df_p = \mathrm{Id}$.

Theorem 1. *Every topologically transitive C^4 nondegenerate almost Anosov diffeomorphism on a 2-dimensional manifold M admits either an SRB measure or an infinite SRB measure.*

Corollary 1. *If μ is an SRB measure, then $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i x} \to \mu$ for m-a.e. x. On the other hand, for infinite SRB measures with $S = \{p\}$, we have $\frac{1}{n}\sum_{i=0}^{n-1}\delta_{f^i x} \to \delta_p$ for m-a.e. x. For more general singular sets S, for any open neighborhood U of S,*

$$\lim_{n\to\infty} \frac{1}{n}\#\left\{k : f^k x \in U,\ 0 \le k \le n-1\right\} = 1 \quad m\text{-a.e. } x \in M$$

As is typical with nonuniformly hyperbolic systems, we denote the *local stable and unstable manifolds* $W^s_\varepsilon(x)$ and $W^u_\varepsilon(x)$ in the usual way:

$$W^u_\varepsilon(x) = \left\{y \in M : d\left(f^{-n}y, f^{-n}x\right) \le \varepsilon\ \forall n \ge 0\right\}$$

and $W^s_\varepsilon(x)$ is defined similarly. It turns out both $W^u_\varepsilon(p)$ and $W^s_\varepsilon(p)$ are differentiable, with $\{p\} = S$. But we make a further assumption for a classification theorem, which is that $W^u_\varepsilon(p)$ and $W^s_\varepsilon(p)$ are C^4 curves.

Lemma 1. *Under these assumptions (nondegeneracy, $S = \{p\}$ is a fixed point, $Df_p = \mathrm{Id}$), $D^2 f_p = 0$.*

So there's a coordinate system so that in some neighborhood of p, we can write f as

$$f(x,y) = \Big(\ x\big(1+\varphi(x,y)\big),\ y\big(1-\psi(x,y)\big)\ \Big)$$

for $(x,y) \in \mathbb{R}^2$, with

$$\begin{aligned}\varphi(x,y) &= a_0x^2 + a_1xy + a_2y^2 + O(|(x,y)|^3),\\ \psi(x,y) &= b_0x^2 + b_1xy + b_2y^2 + O(|(x,y)|^3)\end{aligned}$$

Theorem 2. *Let f be a topologically transitive almost Anosov diffeomorphism on M with $S = \{p\}$. Assuming $Df_p = \mathrm{Id}$ and $W^u_\varepsilon(p)$, $W^s_\varepsilon(p)$ are C^4 curves, with the above coordinate system, we have:*

(i) If $\alpha a_2 > 2b_2$ for some $0 \le \alpha \le 1$, $a_1 = b_1 = 0$, and $a_0b_2 - a_2b_0 > 0$, then f admits an SRB measure.

(ii) If $2a_2 < \alpha b_2$ for some $0 \le \alpha \le 1$ and $a_1b_1 \ne 0$, then f admits an infinite SRB measure.

By nondegeneracy, $a_0, a_2, b_0, b_2 > 0$. So the conditions in part (i) of this theorem imply $\varphi > \psi$ near p. On the other hand, the conditions in part (ii) imply $\psi > \varphi$ near the y axis in some quadrants.

As mentioned, if one of the eigenvalues of Df_p is < 1 and the other is $= 1$, then there is an infinite SRB measure; this suggests that if expansion is "weaker" than contraction, then we get infinite SRB. So our classification theorem confirms this trend: if $\varphi > \psi$, then expansion is stronger, and we get SRB probability measures. But if $\varphi < \psi$, then expansion is weaker, and we have infinite SRB measures.

2 Existence of SRB measures

A major component of the argument for the existence of SRB measures involves showing (M, f) has local product structure at the origin. Therefore, we can talk about *rectangles* (in the Markov partition sense), that is, closed sets $R \subseteq M$ such that whenever $y, z \in R$, $[y,z] := W^u_\varepsilon(y) \cap W^s_\varepsilon(z) \in R$.

Notation: Suppose γ^u and γ^s are segments of W^u and W^s-leaves respectively. Then we denote the rectangle:

$$[\gamma^s, \gamma^u] := \{[y,z] : y \in \gamma^s,\ z \in \gamma^u\}$$

(Note: [Hu] defines this as $[\gamma^u, \gamma^s]$ instead, but I'm reasonably sure this is a typo; defined in this way, it seems $[\gamma^u, \gamma^s]$ would only have one point, clearly not a meaningful rectangle.)

Lemma 2. *There exist curve segments $\gamma^s \subseteq W^s(p)$ and $\gamma^u \subseteq W^u(p)$, with p in the interiors of both γ^u and γ^s, so that the rectangle $P = [\gamma^s, \gamma^u]$ satisfies:*

- *$\exists$ compact $\widehat{W}^s \subseteq W^s(p)$ with $f\widehat{W}^s \subseteq \widehat{W}^s$, $\iota^s \partial^s P \subseteq \widehat{W}^s$, and $\left(\widehat{W}^s \setminus W^s(p,P)\right) \cap \mathring{P} = \varnothing$;*
- *$\exists$ compact $\widehat{W}^u \subseteq W^u(p)$ with analogous properties (with f^{-1} instead of f).*

Moreover, γ^u and γ^s may be chosen so P has arbitrarily small diameter.

This follows from density of $W^u(p)$ and $W^s(p)$.

Lemma 3. *There is a g-invariant Borel probability measure $\overline{\mu}$, where $g = f^\tau : M \setminus P \to M \setminus P$ and $\tau = \tau(x)$ is the first return time for $x \in M \setminus P$, so that $\overline{\mu}$ has absolutely continuous conditional measures on the unstable manifolds of f.*

Proof. To construct $\overline{\mu}$, let P^+ be a component of $fP \setminus P$, $x \in P^+$, and $L = W^u(x, P^+) = W^u(x) \cap P^+$. Let m_L be the Lebesgue measure on L, and let $\overline{\mu}$ be any weak* accumulation point of $\displaystyle\frac{1}{n}\sum_{i=0}^{n-1} g^i_* m_L$.

Absolute continuity follows from the following distortion estimate: $\exists \delta > 0, J_u > 1$ depending on P, so that if γ is a W^u-seg. with $\ell(\gamma) \le \delta$, $\gamma \cap P = \varnothing$, then $\forall x, y \in \gamma$ and $\forall n > 0$,

$$J_u^{-1} \le \frac{\left|Df_y^{-n}|_{E^u_y}\right|}{\left|Df_x^{-n}|_{E^u_x}\right|} \le J_u.$$

This follows (nontrivially) from what Hu refers to a the *local Hölder condition*: $\exists H > 0, \theta > 0, r^* > 0$ so that $\forall x \in M \setminus \{p\}$,

$$d\left(E^u_x, E^u_y\right) \le \frac{H}{\rho_x^{3\theta}} d(x,y)^\theta \qquad \forall y \in B(x, \rho_x^3),$$

where $\rho_x = \min\{|x|, r^*\}$.

Proving this lemma is the major obstacle presented by almost Anosov maps as opposed to Anosov maps. The idea is to take cones $\widetilde{C}^u$ and $\widetilde{C}^s$, in which live the eigenvectors of $D^3 f_p(x, x, \cdot)$, and to define a coordinate system using these cones and various estimates involving the unstable submanifolds. One then shows in this coordinate system, Df_p contracts angles between vectors in $\widetilde{C}^u$. $\square$

To construct the (possibly infinite) SRB measure, let $Q_0 = M \setminus P$, $Q_i = \{x \in M \setminus P : fx, \ldots, f^i x \in P\}$ $\forall i \ge 1$, and define

$$\mu = \sum_{i=0}^\infty f^i_*\left(\overline{\mu}|_{Q_i}\right)$$

Since $\overline{\mu}$ has absolutely continuous invariant measures on unstable manifolds, so does μ. If this series converges, $\mu(M) < \infty$, and we normalize to SRB.

OTOH: If the series diverges, μ is σ-finite. Indeed, if $U \subseteq P$ is any open set containing p, then there is some large n for which $M \setminus U \subseteq M \setminus \Big(\bigcap_{i=-n}^{n} f^i P\Big)$. This set has nonzero measure for only finitely many $f^i_*\left(\overline{\mu}|_{Q_i}\right)$, so $\mu(M \setminus U) < \infty$. So μ is an infinite SRB measure.

One can use Hopf's argument to demonstrate that $(M \setminus P, \overline{\mu}, g)$ is ergodic. It's straightforward to extend this argument to show (M, μ, f) is ergodic for $\mu(M) < \infty$.

We would also like to develop some "ergodic" properties when $\mu(M) = \infty$; namely that $\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i x} \to \delta_p$. First, we need to demonstrate that the stable W^s foliation is locally Lipschitz away from p.

More precisely, for any rectangle $P = [W_r^u(p), W_r^s(p)]$, we need to show there are constants $L > 0$ and $\varepsilon > 0$ such that $\forall x \in M \setminus P$, with $[W_\varepsilon^u(x), W_\varepsilon^s(x)] \cap P \neq 0$, and $\forall z \in W_\varepsilon^s(x)$, the sliding map $\iota : W_\varepsilon^u(x) \to W^u(z)$ is Lipschitz with Lipschitz constant L. This follows from the continuity of $x \mapsto E_x^u, E_x^s$ away from p coupled with the distortion estimates

$$J_u^{-1} \leq \frac{\left|Df_y^{-n}|_{E_y^u}\right|}{\left|Df_x^{-n}|_{E_x^u}\right|} \leq J_u, \qquad y, z \in \gamma \subset W^u(x) \setminus P,\ x \notin P;$$

and

$$J_s^{-1} \leq \frac{\left|Df_y^{-n}|_{E_y^u}\right|}{\left|Df_x^{-n}|_{E_x^u}\right|} \leq J_s, \qquad y, z \in \gamma \subset W^s(x) \setminus P,\ x \notin P.$$

i.e. if γ is a seg. of an unstable leaf $W_\varepsilon^u(x)$, one can show $\ell(\iota\gamma) \leq L' J_s J_u \ell(\gamma)$, for some L'.

To prove the desired weak* convergence, we show that given small $\alpha > 0$, $\varepsilon > 0$, there are neighborhoods $P_2 \subseteq P_1$ of p with $\mathrm{diam} P_1 \leq \alpha$ such that for m-a.e. $x \in M \setminus P_2$,

$$\frac{\#\left\{k : f^k x \in M \setminus P_1,\ 0 \leq k \leq n\right\}}{\#\left\{k : f^k x \in P_1 \setminus P_2,\ 0 \leq k \leq n\right\}} < \varepsilon.$$

That is, a.e. $x \in M$ spends almost 100% of its time in an annulus of small radii around p. To do this, we let $P_1 = [W_\beta^s(p), W_\beta^u(p)]$ for a small β, and let P_2 be small enough so that $\mu(M \setminus P_1) \leq \varepsilon\mu(P_1 \setminus P_2)$. One can now prove invariance and ergodicity of the map $g_2 : (M \setminus P_2, \mu_2) \to (M \setminus P_2, \mu_2)$, where $\mu_2 = \mu|_{M\setminus P_2}$. Applying the Birkhoff Ergodic Theorem to $(M \setminus P_2, \mu_2, g_2)$ gives the desired result.

This again makes sense in the context of the notion that "weaker expansion $\implies$ infinite SRB". Indeed, in this case, m-a.e. points in M spend almost all of their time near the singular fixed point p.

3 Unstable submanifolds

The major piece of the analysis that makes this notably harder than with fully Anosov systems comes in proving basic topological properties of $W^u(x)$ and $W^s(x)$-foliations. This boils down to performing certain estimates on the linear map $\mathrm{Id} + \frac{1}{2}D^3 f_p(x, x, \cdot) : T_pM \to T_pM$.

Proposition 1. *There is an invariant decomposition of the tangent bundle into $TM = E^u \oplus E^s$ such that $E_x^\eta \subset \mathcal{C}_x^\eta$ and $Df_x E_x^\eta = E_{fx}^\eta$, $\eta = s, u$. Except at the singular fixed point p, the decomposition is continuous.*

Proof. The construction of $TM = E^s \oplus E^u$ is standard on $M \setminus \{p\}$, as is the proof of continuity. For $E_p^s \oplus E_p^u$, the construction is more involved.

Lemma 4. *There are constants $0 < \widetilde{r} \leq r_0$, $0 < \widetilde{\kappa}^u \leq \kappa^u$ and $0 < \widetilde{\kappa}^s \leq \kappa^s$, cones $\widetilde{\mathcal{C}}^u$ and $\widetilde{\mathcal{C}}^s$ such that $\forall x \in B(p, \widetilde{r})$, $\widetilde{\mathcal{C}}^\eta \supseteq \mathcal{C}_x^\eta$ $(\eta = s, u)$ and*

$$|Df_x v| \geq \left(1 + \widetilde{\kappa}^u d(x, S)^2\right)|v| \quad \forall v \in \widetilde{\mathcal{C}}_x^u,$$
$$|Df_x v| \leq \left(1 - \widetilde{\kappa}^s d(x, S)^2\right)|v| \quad \forall v \in \widetilde{\mathcal{C}}_x^s.$$

To show this, let $A_x = \frac{1}{2} D^3 f_p(x, x, \cdot)$, and denote

$$\mathcal{C}_x(\beta) = \left\{ v \in \mathbb{R}^2 : \langle v, A_x v \rangle \geq \beta |x|^2 |v|^2 \right\}$$

One can use nondegeneracy of f and the Taylor expansion $Df_x = \mathrm{Id} + A_x + R_F(x)$ (where $\|R_F(x)\| = O(|x|^3)$), since $D^2 f_x = 0$, to show

$$\widetilde{\mathcal{C}}^u := \bigcap_{x \in \mathbb{S}^1} \mathcal{C}_x\big(\frac{3}{5}\kappa^u\big) \supsetneq \mathcal{C}_p^u$$

and that $|Df_x v| \geq \left(1 + \frac{1}{2}\kappa^u |x|^2\right) |v| \ \forall v \in \widetilde{\mathcal{C}}^u$.

One can then show that for $a \in \mathbb{R}^2 \setminus \{0\}$, the linear map $\mathrm{Id} + \frac{1}{2} D^3(a, a, \cdot)$ has an eigenvector in $\widetilde{\mathcal{C}}^u$ and an eigenvector in $\widetilde{\mathcal{C}}^s$. (Argument is pretty much just pages of elementary geometry.)

There are a few more steps, but the final one is to show that there is a unique subspace $E^+ \subseteq \mathbb{R}^2$ such that $\forall a \in E^+ \setminus \{0\}$, a is an eigenvector of $\frac{1}{2} D^3 f_p(a, a, \cdot)$ with positive eigenvalue. E^- can analogously be constructed using f^{-1}. We set $E_p^u = E^+$ and $E_p^s = E^-$. □

Proposition 2. *$\forall x \in M$, $W_\varepsilon^u(x)$ is a curve tangent to E_x^u*

Proof. For $x \neq p$, the argument is standard. For $x = p$, Hu's strategy is to let $\Omega \subset B(p, \varepsilon)$ be those points that can be joined to p by a curve tangent to vectors in $\widetilde{\mathcal{C}}^u$, and use various estimates related to the fact that

$$W_\varepsilon^u(p) = \bigcap_{i=0}^{\infty} \left(f^i \Omega \cap B(p, \varepsilon) \right).$$

□

Note that $x \mapsto E_x^\eta$ is not continuous at $x = p$. Despite this, the end result of this cone analysis gives us that $E_x^\eta \subseteq \mathcal{C}_x^\eta$, giving us local product structure at p.

4 Other statistical properties

There are other statistical properties of almost Anosov and almost hyperbolic systems under current investigation. Many of them rely on the existence of a Markov partition of (M, f).

Typically Hu and others have either assumed the existence of a Markov partition, or have assumed almost Anosov maps are as a rule topologically conjugate to Anosov diffeomorphisms (in which case the existence of a Markov partition is immediate).

Given that almost Anosov diffeomorphisms have local product structure, perhaps one can construct a Markov partition using similar strategies to Bowen without needing Anosov conjugacy. I have not yet tried this.

Theorem 3. *A nondegenerate almost Anosov diffeomorphism $f : \mathbb{T}^2 \to \mathbb{T}^2$ with finitely many fixed points, and which is linear Anosov outside of a neighborhood of the origin and has the standard nondegenerate form within the origin, is topologically conjugate to an Anosov diffeomorphism.*

Proof. We need to show that f is both expansive and is the limit of a sequence of Anosov diffeomorphisms. For $x, y \in \mathbb{T}^2 \setminus \{0\}$, the argument for expansiveness is standard: suppose for every $\varepsilon > 0$ we have distinct $x, y \in \mathbb{T}^2$ so that $d(f^n x, f^n y) < \varepsilon$ for every $n \in \mathbb{Z}$. Then $y \in W_\varepsilon^u(x) \cap W_\varepsilon^s(x)$. If we assume $\varepsilon < \min\{\delta_0, \varepsilon_0\}$, so that $W_{\varepsilon_0}^\eta(x) \supseteq W_\varepsilon^\eta(x)$ for $\eta = s, u$, we get

$$y \in W_\varepsilon^u(x) \cap W_\varepsilon^s(x) \subseteq W_{\varepsilon_0}^u(x) \cap W_\varepsilon^s(x) = \{x\}.$$

Therefore $y = x$, contradiction. Let ε be the expansiveness constant for $x, y \in \mathbb{T}^2 \setminus \{0\}$. Now suppose f fails to be expansive at the origin, i.e. suppose that for every $\delta > 0$ there is an $x \in \mathbb{T}^2$ so that $d(f^n x, 0) < \delta$

for every $n \in \mathbb{Z}$. Take $\delta < \varepsilon/2$, and without loss of generality, assume δ is small enough so that $B_\delta(0)$ contains no fixed points besides 0. Then let $x \in B_\delta(0)$ be so that $d\left(f^n x, 0\right) < \delta$ for every $n \in \mathbb{Z}$. Then, $d\left(f^n x, f^{n+1} x\right) \leq d\left(f^n x, 0\right) + d\left(f^{n+1} x, 0\right) < \varepsilon$ for every $n \in \mathbb{Z}$. Since $fx \neq x$ for small δ, this contradicts our previous conclusion that f is expansive on $\mathbb{T}^2 \setminus \{0\}$. So $f : \mathbb{T}^2 \to \mathbb{T}^2$ is expansive.

To prove that f is a limit of a sequence of Anosov diffeomorphisms, we define a homotopy $H : \mathbb{T}^2 \times [0,1] \to \mathbb{T}^2$ so that $H_0 = f$, and H_ε is Anosov for every $\varepsilon \in [0,1]$. For $r_0 > 0$ small, the disc B_{r_0} is a coordinate chart in which f has the form

$$f(x,y) = \Big(\ x\big(1+\varphi(x,y)\big),\quad y\big(1-\psi(x,y)\big)\ \Big), \tag{1}$$

for $(x,y) \in \mathbb{R}^2$ and

$$\begin{aligned} \varphi(x,y) &= ax^2 + by^2 + O\left(|(x,y)|^3\right), \\ \psi(x,y) &= cx^2 + dy^2 + O\left(|(x,y)|^3\right). \end{aligned}$$

The differentials in B_{r_0} are of the form

$$Df(x,y) = \begin{pmatrix} 1 + 3ax^2 + by^2 & 2bxy \\ -2cxy & 1 - cx^2 - 3dy^2 \end{pmatrix} + \mathcal{O},$$

where $\mathcal{O}$ is a matrix of terms of order $|(x,y)|^3$. By the cone decomposition of the tangent spaces of points other than the origin, these matrices are hyperbolic. Since hyperbolicity is an open condition on matrices, assuming r_0 is sufficiently small, we may assume $\mathcal{O} = 0$. Moreover, by openness of hyperbolicity, there are continuous functions $\pi, \rho : \mathbb{T}^2 \to (0,\infty)$ for which the matrix

$$e\begin{pmatrix} 1 + (3a - \alpha)\,ax^2 + by^2 & 2bxy \\ -2cxy & 1 - cx^2 - (3d - \beta)y^2 \end{pmatrix} \tag{2}$$

is hyperbolic for $0 \leq \alpha \leq \pi(x,y)$ and $0 \leq \beta \leq \rho(x,y)$. Define the functions $\alpha, \beta : [0,1] \to [0,1]$ by

$$\alpha(s) = \inf_{x^2+y^2=s^2} \pi(x,y) \quad \text{and} \quad \beta(s) = \inf_{x^2+y^2=s^2} \rho(x,y).$$

Now define the continuous maps $g_\varepsilon, h_\varepsilon : [0,1] \to [0,1]$ for each $\varepsilon \in [0, r_0]$ so that:

(i) $g_\varepsilon(t) = h_\varepsilon(t) = 0$ for $t \geq \varepsilon^2$;

(ii) $g_\varepsilon \to 0$ and $h_\varepsilon \to 0$ in C^1 as $\varepsilon \to 0$;

(iii) for $s < \varepsilon$, we have $-\frac{1}{2}\alpha(s) < g'_\varepsilon(s^2) < 0$ and $-\frac{1}{2}\beta(s) < h'_\varepsilon(s^2) < 0$.

Note (ii) and (iii) tell us $g_\varepsilon(t) > 0$ and $h_\varepsilon(t) > 0$ for $t < \varepsilon^2$. Let $H_\varepsilon : \mathbb{T}^2 \to \mathbb{T}^2$ be maps for each $\varepsilon > 0$ so that in the coordinate ball B_{r_0}, H_ε is of the form

$$H_\varepsilon(x,y) = \Big(x\left(1 + g_\varepsilon\left(x^2+y^2\right) + ax^2 + by^2\right),\quad y\left(1 - h_\varepsilon\left(x^2+y^2\right) - cx^2 - dy^2\right)\Big),$$

and we further assume that outside of B_{r_1}, the map $H_t \equiv F$ is linear Anosov for all t, and in the annulus $B_{r_1} \setminus B_{r_0}$, H_t smooths out to F. By assumptions (i) and (ii) of g_ε, f is the C^1 limit of H_ε as $\varepsilon \to 0$, so we only need to show each H_ε is Anosov for $\varepsilon \in (0, \varepsilon_0)$ for some small ε_0. To that end, the derivative of $H_\varepsilon : \mathbb{T}^2 \to \mathbb{T}^2$ for a fixed ε is

$$DH_\varepsilon(x,y) = \begin{pmatrix} 1 + \Phi_\varepsilon(x,y) & 2xy\left(g'_\varepsilon\left(x^2+y^2\right) + b\right) \\ -2xy\left(h'_\varepsilon\left(x^2+y^2\right) + c\right) & 1 - \Psi_\varepsilon(x,y) \end{pmatrix} \tag{3}$$

where

$$\begin{aligned} \Phi_\varepsilon(x,y) &= g_\varepsilon\left(x^2+y^2\right) + \big(3a + 2g'_\varepsilon\left(x^2+y^2\right)\big)x^2 + by^2, \\ \Psi_\varepsilon(x,y) &= h_\varepsilon\left(x^2+y^2\right) + \big(3d + 2h'_\varepsilon\left(x^2+y^2\right)\big)y^2 + cx^2. \end{aligned}$$

Our objective is to show that these linear maps are hyperbolic for every $(x, y) \in \mathbb{T}^2$. Since $H_\varepsilon(x, y) = F$ for $(x, y) \notin B_\varepsilon$, and we know DF is hyperbolic everywhere, we only need to check the case when $x^2 + y^2 < \varepsilon^2$.

Hyperbolicity may fail in two ways: if $\Phi_\varepsilon(x, y)$ or $\Psi_\varepsilon(x, y)$ are too small at some point (x, y), or if the upper right and lower left terms of (3) are too far from 0. The latter concern is easy to address: assuming $|(x, y)|$ is small, since $\alpha\left(x^2 + y^2\right) \to 0$ and $\beta\left(x^2 + y^2\right) \to 0$ as $|(x, y)| \to 0$, condition (iii) on the definition of g_ε and h_ε give us $g'_\varepsilon\left(x^2 + y^2\right) > -b$ and $h'_\varepsilon(\left(x^2 + y^2\right) > -c$. In particular, the furthest either the upper right or lower left entries of (3) can be from 0 are $2bxy$ and $-2cxy$ for any x, y.

To address the first concern, i.e. ensuring $\Phi_\varepsilon(x, y)$ and $\Psi_\varepsilon(x, y)$ are not too small, we observe:

$$\Phi_\varepsilon(x, y) \geq g_\varepsilon\left(x^2 + y^2\right) + \big(3a - \alpha\left(|(x, y)|\right)\big)x^2 + by^2 > \big(3a - \pi(x, y)\big)x^2 + by^2,$$

and

$$\Psi_\varepsilon(x, y) \geq h_\varepsilon\left(x^2 + y^2\right) + \big(3d - \beta(|(x, y)|)\big)y^2 + cx^2 > \big(3d - \rho(x, y)\big)y^2 + cx^2.$$

Therefore, the furthest each linear map $DH_\varepsilon(x, y)$ could possibly be from being hyperbolic would be if $DH_\varepsilon(x, y)$ were of the form

$$\begin{pmatrix} 1 + \big(3a - \pi(x, y)\big)x^2 + by^2 & 2bxy \\ -2cxy & 1 - \big(3d - \rho(x, y)\big)y^2 - cx^2 \end{pmatrix}$$

and as we saw in (2), this matrix is still hyperbolic.

$\square$

Many arguments giving statistical properties of almost Anosov diffeomorphisms rely on quotienting the manifold by unstable leaves in each Markov partition rectangle, and observing analogous properties of the resulting *almost expanding* map.

Definition 3. A map $f : M \to M$ is *almost expanding* on a closed invariant subset $\Lambda \subseteq M$ if there is some finite set $S \subseteq M$ such that $\forall \varepsilon > 0\ \exists \kappa > 0$ s.t.

$$|Df_x v| > \kappa |v| \qquad \forall x \in M \setminus B(S, \varepsilon), \quad \forall v \in T_x M.$$

Manneville-Pomeau (sp?) map comes to mind.

Recall our coordinate system-based classification theorem for almost Anosov maps:

$$f(x, y) = \Big(\ x\big(1 + \varphi(x, y)\big),\ y\big(1 - \psi(x, y)\big)\ \Big)$$

for $(x, y) \in \mathbb{R}^2$, with

$$\begin{aligned} \varphi(x, y) &= a_0 x^2 + a_1 xy + a_2 y^2 + O(|(x, y)|^3), \\ \psi(x, y) &= b_0 x^2 + b_1 xy + b_2 y^2 + O(|(x, y)|^3) \end{aligned}$$

Theorem. *Let f be a topologically transitive almost Anosov diffeomorphism on M with $S = \{p\}$. Assuming $Df_p = \mathrm{Id}$ and $W^u_\varepsilon(p)$, $W^s_\varepsilon(p)$ are C^4 curves, with the above coordinate system, we have:*

(i) If $\alpha a_2 > 2b_2$ for some $0 \leq \alpha \leq 1$, $a_1 = b_1 = 0$, and $a_0 b_2 - a_2 b_0 > 0$, then f admits an SRB measure.

(ii) If $2a_2 < \alpha b_2$ for some $0 \leq \alpha \leq 1$ and $a_1 b_1 \neq 0$, then f admits an infinite SRB measure.

Theorem 4 (Hu)**.** *An almost Anosov map with $\alpha a_2 > 2b_2$, $a_1 = b_1 = 0$, and $a_0 b_2 > a_2 b_0$ (which, as we recall, admits an SRB probability measure μ) has polynomial decay of correlations w.r.t. Lipschitz functions, with degree $\frac{a_2}{2b_2} - 1$. That is, for Lipschitz $g, \widehat{g}$, $\exists C = C(g, \widehat{g})$ s.t.*

$$\left| \int g \cdot (\widehat{g} \circ f^n)\, d\mu - \int g\, d\mu \int \widehat{g}\, d\mu \right| \leq C n^{-\frac{a_2}{2b_2} + 1} \qquad \forall n \geq 1.$$

Proof. Boils down to checking bounds on unstable leafs of f in different sets

$$P'_n = \{x \notin P : f^i x \in P \;\forall 1 \le i \le n\}$$

which follows since $\dfrac{d\mu_u \circ f}{d\mu_u}(x)$ is proportional to $\mu(P_{n-1})/\mu(P_n) = \mu(P'_{n-1})/\mu(P'_n)$ for all n, for all $x \in P_n$, where μ_u = conditional W^u-measure. □

More recently, Hu and Zheng (his student) proved a somewhat stronger result, with some additional assumptions on the diffeomorphism.

Uses fact that if $f : \mathbb{T}^2 \to \mathbb{T}^2$ is almost Anosov, then for any nbhd U of p, $\exists \theta^* \in (0,1)$ for which the unstable subspaces in U are Hölder continuous with Hölder exponent θ^*.

Theorem 5 (Hu, Zheng)**.** *Let f be a C^r $(r \ge 4)$ topologically mixing nondegenerate almost Anosov diffeomorphism with an indifferent fixed point p. Suppose $a_0b_2 > a_2b_0$, $4b_2 < a_2$, and $a_1 = b_1 = 0$. Fix $\alpha, \beta \in (0, 1/2)$ with*

$$\frac{\alpha}{1+\alpha} < \beta < \frac{2a_2b_2}{a_2^2 + a_2b_2 + b_2^2} < \frac{2b_2}{a_2} < \alpha.$$

Then for any neighborhood U of p, and any Hölder continuous Φ and Ψ with exponent θ and $\mathrm{supp}\Phi, \mathrm{supp}\Psi \subseteq M \setminus U$*, and $\int \Phi \, d\mu \int \Psi, d\mu \neq 0$, we have*

$$\frac{A'}{n^{\frac{1}{\beta}-1}} \le \left| \mathrm{Cor}_n(\Phi, \Psi, f, \mu) \right| \le \frac{A}{n^{\frac{1}{\alpha}-1}},$$

where μ is an SRB measure, $\theta \in \big(\max\{(1/\beta - 1/\alpha)(3/2 + b_0/(2a_0))^{-1}, \theta^*\}, 1\big]$, and $A, A' > 0$ are constants depending on Φ and Ψ.

Proof uses "renewal theory", which I am not familiar with in large part but I'll look into that.

5 Some open questions

- Can this analysis be applied to higher-dimensional systems?
- Can nondegeneracy be relaxed?
- Can this be adapted to almost hyperbolic attractors?
- What if Df_p is indifferent and nondiagonalizable?
  - I did find a paper (Catsigeras & Enrich, 2000) addressing this question. It seems to suggest that such systems do have SRB measures under certain conditions on their Taylor polynomials. For example:
  $$f(x,y) = \Big(x + ay + r(x,y),\; y + s(x,y) \Big)$$
  where $a \neq 0$ and r, s are $O(|(x,y)|^2)$ (namely if the coeff. of x^2 in r is 0 and coeff. of x^3 in s is $\neq 0$, this system admits an SRB measure).

Another property of many dynamical systems is *stochastic stability*. A random perturbation $\mathcal{F}$ of $f : M \to M$ is essentially a Markov chain with states space M and transition probabilities $\mathbb{P}(\cdot|x)$. A measure μ is *invariant for $\mathcal{F}$* if for any Borel $E \subseteq M$,

$$\mu(E) = \int \mathbb{P}(E|x)\, d\mu(x)$$

The system (M, f, μ_0) is *stochastically stable* if given a 1-parameter family of random perturbations $\{\mathcal{F}_\varepsilon : \varepsilon > 0\}$ with $\mathcal{F}_\varepsilon \to f$ (i.e. $\mathbb{P}_\varepsilon(\cdot|x) \to \delta_{fx}$ weakly $\forall x$), the invariant measures μ_ε of $\mathcal{F}$ converge $\mu_\varepsilon \to \mu_0$ weakly as $\varepsilon \to 0$.

It is known that Anosov maps (and hyperbolic maps generally) with SRB measures are stochastically stable.

- Under what families of random perturbations (if any) are almost hyperbolic systems (Λ, μ, f) stochastically stable?

There are other questions pertaining more broadly to the thermodynamics of such maps, e.g. the subject of my research.

- Under what conditions do there exist equilibrium states for geometric t-potentials $\varphi_t(x) = -t \log \left| Df_x|_{E^u_x} \right|$?

Conjecture 1. *The answer to the last question is yes for $M = \mathbb{T}^2$, $p = 0$, and f having an SRB probability measure in the first place.*

The objective here is to construct a Young tower whose base is a Markov partition element away from the singularity at 0. At the moment, this involves proving the following:

1. For $x \in \Lambda^s_i$, define $\tau(x) = \tau_i$ to be the inducing time, and the induced map $F : \bigcup_{i\in\mathbb{N}} \Lambda^s_i \to \Lambda$ by $F|_{\Lambda^s_i} = f^{\tau_i}|_{\Lambda^s_i}$. Then there is $0 < a < 1$ s.t. for any $i \in \mathbb{N}$, we have:
   - For $x \in \Lambda^s_i$, $y \in \gamma^s(x)$,
   $$d(F(x), F(y)) \le a d(x, y);$$
   - For $x \in \Lambda^s_i$, $y \in \gamma^u(x) \cap \Lambda^s_i$,
   $$d(x, y) \le a d(F(x), F(y)).$$

   That is, we need that the induced map contracts points on the same stable leaf in forward time, and points on the same unstable leaf in backwards time;

2. Let $J^u F(x) = \det \left| DF|_{E^u_x} \right|$. There exists $c > 0$ and $\kappa \in (0, 1)$ such that:
   - For all $n \ge 0$, $x \in F^{-n}\left(\bigcup_{i\ge1} \Lambda^s_i\right)$, and $y \in \gamma^s(x)$, we have
   $$\left| \log \frac{J^u F(F^n(x))}{J^u F(F^n(y)} \right| \le c\kappa^n.$$
   - For any $i_0, \dots, i_n \in \mathbb{N}$, $F^k(x)$, $F^k(y) \in \Lambda^s_{i_k}$ for $0 \le k \le n$ and $y \in \gamma^u(x)$, we have
   $$\left| \log \frac{J^u F(F^{n-k}(x))}{J^u F(F^{n-k}(y)} \right| \le c\kappa^k.$$

This is similar to what is done by Hu, Zhang, and Young, although they don't use the phrase "Young tower".