Research Statement

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My general area of research is dynamical systems. Fundamentally, a dynamical system is a triple (X, G, f), where:

- X is a *state space*, or a set with some underlying structure (typically either a topological space, measure space, or smooth manifold);
- G is an abelian group or monoid (typically either $\mathbb{N}$, $\mathbb{Z}$, or $\mathbb{R}$); and,
- $f : X \times G \to X$ is a function (typically continuous or measurable as appropriate), written $f(x,t) = f^t(x)$, so that $f^{s+t} = f^s \circ f^t$ for $s, t \in G$, and $f^0 = \text{Id}$.

Historically, modern questions in dynamical systems began with the study of differential equations, studying phenomena such as asymptotic stability and periodic orbits of solutions to differential equations. One of the earliest differential equations problems that motivated dynamical systems of today is Isaac Newton's three-body problem. Other historical examples of dynamical systems include fluid mechanics and the Navier-Stokes equations; the logistic growth model of population dynamics; and models of atmospheric convection such as the Lorenz attractor.

These physical problems have motivated a wider investigation into abstract dynamical systems. The two broadest categories of dynamical systems are *continuous* dynamics, in which the group is $\mathbb{R}$ and the function f is a flow on the space X; and *discrete* dynamics, in which the group or monoid is typically $\mathbb{Z}$ or $\mathbb{N}$ (although there are also examples where the group is instead $\mathbb{Z}^n$). Continuous dynamics includes systems of differential equations, and in pure mathematics, geodesic flows on Riemannian manifolds. Discrete dynamics, on the other hand, includes symbolic dynamics (in which the state space is the set of one- or two-sided infinite sequences on a countable alphabet, and the map shifts the sequence to the left by one entry), complex dynamics (in which the state space is $\mathbb{C}^n$ and the map is a complex-analytic function, leading to phenomena such as Julia and Mandelbrot sets), and interval exchange transformations (in which the state space is the unit interval partitioned into subintervals, and the map permutes the subintervals), among several other examples.

Within discrete dynamical systems is the field of *ergodic theory*. Ergodic theory studies dynamical systems where the state space X is a probability space and the transformation $T : X \to X$ is measurable. Typically, an ergodic system is endowed with a measurable function $\varphi : X \to \mathbb{R}$, called an *observable*, so that the sequence of functions $\{T^n \circ \varphi\}_{n \geq 0}$ on X form a stochastic process. In this way, ergodic theory addresses questions in probability theory using techniques from dynamical systems. Its applications include identifying the mathematical conditions under which the average observed data in a series of experimental trials asymptotically reflects the true behavior of the subject of the experiment as the number of trials increases (this is known as the Birkhoff ergodic theorem). This is one way in which ergodic theory has proven to be a productive tool in experimental design and data analysis.

In addition to ergodic theory, another crucial area of dynamical systems used in my research is smooth dynamics, and in particular, hyperbolic dynamics. Hyperbolic dynamics can be traced to Henri Poincaré's progress on the three-body problem, when he discovered that solutions to the three-body problem are highly sensitive to the initial conditions of the system. This led to the development of the study of what are now referred to as "chaotic" dynamical systems, or dynamical systems (X, f) whose orbits $\{f^n(x_0)\}$ change significantly depending on the orbit's initial point x_0 in the state space X. In the smooth category, these are realized by what are known as hyperbolic smooth dynamical systems. These include Anosov and pseudo-Anosov diffeomorphisms, Smale horseshoes, and Smale-Williams solenoids. They also include symbolic dynamical systems such as Bernoulli shifts and positive-entropy subshifts of finite type, which can be shown to be measure-theoretically isomorphic to smooth systems such as Anosov diffeomorphisms. Within smooth dynamics, one can ask about the measurable and ergodic properties of different smooth dynamical systems, and in particular, whether there are measures that are significant from an ergodic theory perspective but respect the smooth structure of the state space. This area of study is known as *smooth ergodic theory*.

The theory of dynamical systems employs several tools to measure disorder in both topological and measurable dynamical systems, such as ergodicity, rates of mixing, and topological and measurable entropy.

In the 1970s, Rufus Bowen, David Ruelle, Yakov Sinai, and others began using mathematical tools from statistical physics and thermodynamics to further describe ergodic and stochastic properties of smooth dynamical systems. Inspired by the work of physicists such as Gibbs and Helmholtz, and treating observables in ergodic theory as a kind of physical potential, they developed a theory of physically meaningful probability measures for smooth dynamical systems, which are now known as *Gibbs measures* or *Gibbs states*, as well as a mathematical counterpart to Helmholtz free energy, now known as *topological pressure*. Topological pressure has proven to be a valuable tool in the study of Gibbs measures for smooth dynamical systems, and in particular has profound connections to a subclass of Gibbs measures known as *equilibrium states*. The study of Gibbs states, entropy, and topological pressure in dynamical systems is known as *thermodynamic formalism*, and plays a pivotal role in smooth ergodic theory.

My area of research includes both smooth ergodic theory and thermodynamic formalism. I am interested in using the tools of thermodynamics to study statistical properties of nonuniformly hyperbolic dynamical systems. My dissertation studies the existence and uniqueness of equilibrium states for both pseudo-Anosov and almost Anosov surface diffeomorphisms, which includes measures of maximal entropy and physical smooth measures for both systems. In addition, my research has explored several statistical properties of these nonuniformly hyperbolic systems with respect to these equilibrium states. These properties include exponential decay of correlations and the Central Limit Theorem, using a family of geometrically significant observables as our potential functions. However, in many instances, it is known that the decay of correlations for nonuniformly hyperbolic dynamical systems has a polynomial rate, not an exponential rate. One of my current projects involves proving polynomial decay of correlations for pseudo-Anosov diffeomorphisms for a particular geometric potential.